APPLICATIONS OF PONTRYAGIN'S MAXIMUM PRINCIPLE TO ECONOMICS

by

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APPLICATIONS OF PONTRYAGIN'S MAXIMUM PRINCIPLE TO ECONOMICS*

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<u>Lecture I</u>: The problem of consumption-optimal growth illustrated by th one-sector model with linear objective functional. Economic emphasis on the interpretation of auxiliary variables as social demand prices and proof of certain "exact turnpike theorems." Mathematical emphasis on problems where application of the Maximum Principle requires the study of dynamical systems with nonunique flows.

<u>Lecture II</u>: The two-sector model. The nonlinear objective functional. "Inexact turnpike theorems."

<u>Lecture III</u>: Emphasis on the infinite horizon planning problem. The transversality condition. Present discounted value as a linear functional no representable by an improper integral.

<u>Lecture IV</u>: Descriptive models of heterogeneous capital accumulation due t Hahn and Shell-Stiglitz. Nonuniqueness of momentary equilibrium. An unstable dynamical system. Relation to mathematics presented in Lecture Boundedness and the transversality condition.

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LECTURE I

My assignment is to discuss applications of Pontryagin's Maximum Principle to economic theory. The first two of my lectures will be devoted to the problem of consumption-optimal economic growth. The seminal paper in this field is the 1928 contribution of Frank Ramsey.[1] From a mathematical point of view, the Ramsey paper is a straightforward application of the Euler equations of the classical calculus of variations. Because I have been assigned the task of presenting applications of the Maximum Principle, I will have very little directly to say about Ramsey's article and many other important contributions to the optimal growth literature.[2] This is not because I think that there is overriding virtue in using the very latest technique; this is simply the result of the narrowness of my assignment.

We begin by recalling the simple aggregative model of economic growth.[3] In the model economy, there are two factors of production, capital and labor, that cooperate in the production of homogeneous output. At any instant in time a fraction of this homogeneous output may be allocated to investment in capital accumulation. The capital stock is bolted-down in the sense that in itself capital is not fit for consumption. If $K(t)$ and $L(t)$ denote the stocks of capital and labor, respectively, at time t, then the rate of output at time t, $Y(t)$, is given by

$$Y(t) = F[K(t), L(t)] .$$

$F[\cdot]$ is the neoclassical production function, which is twice continuously differentiable, and exhibits constant returns to scale, i.e., $F[\cdot]$ is positively homogeneous of degree one in K and L, yielding that $\theta Y = F[\theta K, \theta L]$ for $K, L \geq 0$ and θ a positive scalar.

Let $C(t) \geq 0$ and $Z(t) \geq 0$ denote the respective rates at time t of consumption and investment; let $0 \leq s(t) \leq 1$ denote the fraction of output at time t which is saved (and invested). Then we have the simple national income identities[4]

$$Y = C + Z = (1 - s)Y + sY .$$

If capital is subject to radioactive decay at the constant relative rate $\mu > 0$, then growth of the capital stock is given by the differential equation

$$\overset{\circ}{K} = sF[K, L] - \mu K .$$

Further assume that labor is inelastically supplied and is growing at the constant relative rate $n > 0$; $\overset{\circ}{L} = nL$.

Denoting per worker quantities by lower case letters, $y \equiv Y/L$, $k \equiv K/L$, etc., yields

$$\overset{\circ}{k} = sf(k) - \lambda k,$$

where $f(k) \equiv F(k, 1)$ and $\lambda \equiv \mu + n$. It is assumed that the marginal product of capital is positive but declining, i.e.,

$$f'(k) > 0 \quad \text{and} \quad f''(k) < 0 \quad \text{for } 0 < k < \infty .$$

In order to make the discussion simple, I also assume that "Inada's limit conditions" are satisfied, i.e.,

$$\begin{cases} \lim_{k \downarrow 0} f(k) = 0, & \lim_{k \uparrow \infty} f(k) = \infty, \\ \lim_{k \downarrow 0} f'(k) = \infty, & \lim_{k \uparrow \infty} f'(k) = 0. \end{cases}$$

In Figure 1, the unique solution to $f(k) = \lambda k$ is denoted by $\tilde{k}$ and is interpreted as the maximum sustainable capital-labor ratio. The unique solution to $f'(k) = \lambda k$ is denoted by $\hat{k}$. In the economic growth literature,[5] $\hat{k}$ is referred to as the Golden Rule capital-labor ratio. The importance of the Golden Rule capital-labor ratio is that it serves as a separatrix between classes of efficient development programs and classes of inefficient development programs, but its original derivation followed from an exceedingly simple <u>Gedankenexperiment</u>: Say that a central planner whose aim is to maximize consumption per worker were able to choose a capital-labor ratio k that he would maintain forever. What capital-labor ratio would he choose? From Figure 1 we see that the planner's feasible choice set is the interval $[0, \tilde{k}]$ and that at $\hat{k}$, $c = f(k) - \lambda k$ is maximized.

Next I will be precise about the normative significance of the Golden Rule capital-labor ratio. I proceed with the definition of inefficiency of development (implying, of course, the definition of efficiency). Then, I will sketch the proof of an Inefficiency Theorem due to Phelps and Koopmans.[6]

Definition: A feasible program of development $\{(c(t), k(t)): t \in [0,\infty)\}$ is said to be inefficient if there exists another program $\{(c^{\dagger}(t), k^{\dagger}(t)): t \in [0, \infty)\}$ such that $k(0) = k^{\dagger}(0)$, $c^{\dagger}(t) \geq c(t)$ for all $t \in [0,\infty)$, and $c^{\dagger}(t) > c(t)$ for some interval $[t', t''] \subset [0,\infty)$.

Theorem: Any feasible program $\{(c(t), k(t)): t \in [0,\infty)\}$ for which there exists a date $t_0 < \infty$ after which

$$k(t) \geq \hat{k} + \varepsilon$$

(where $\varepsilon > 0$ is independent of t) is inefficient.

The proof that is given here requires that capital be freely disposable or at least that it may be set aside and left idle.

Proof: Define a new program $\{(c_*(t), k_*(t)): t \in [0,\infty)\}$ by

$$k_*(t) = \begin{cases} k(t), & t \in [0, t_0], \\ k(t) - \varepsilon, & t \in (t_0, \infty). \end{cases}$$

and

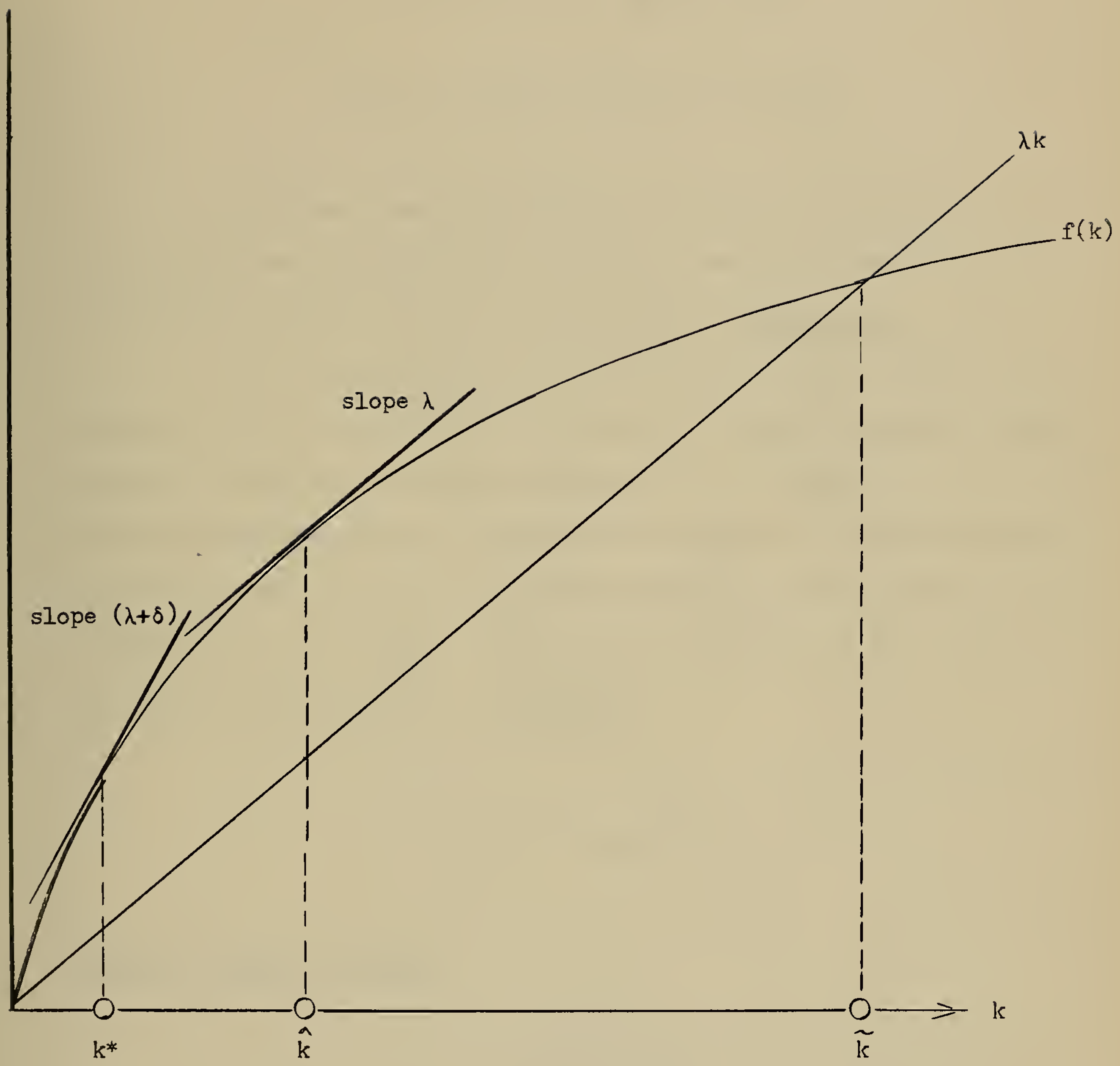


Figure 1

$$c_* = \begin{cases} c = f(k) - \lambda k - \overset{\circ}{k}, & t \in [0, t_0], \\ f(k - \epsilon) - \lambda(k - \epsilon) - \overset{\circ}{k}, & t \in (t_0, \infty). \end{cases}$$

It follows directly from Figure 1 that for $t > t_0$, $c_*(t) > c(t)$. Thus all programs with capital-labor ratios "forever" bounded above the Golden Rule capital-labor ratio are dynamically <u>inefficient</u>.

We have completed most of the preliminary material basic to any analysis of the one-sector model. We turn to a simple one-sector planning problem. Assume that the planning period is the interval $[0, T]$, $0 < T < \infty$. The central planner desires to maximize the integral of discounted consumption per worker[7] subject to technical feasibility, inherited initial factor endowments, and some terminal requirement on supply of factors. Stated specifically, the problem is to <u>maximize</u>

$$\int_0^T (1 - s)f(k)e^{-\delta t}dt,$$

<u>subject to the constraints:</u>

$$\overset{\circ}{k} = sf(k) - \lambda k,$$

$$s \in [0, 1],$$

$$k(0) = k_0, \quad k(T) \geq k_T,$$

where δ, $\lambda \equiv n + \mu$, k_0 and k_T are given positive scalars and $s(t)$ is some measurable control or policy variable to be chosen by the planner. δ is interpreted as the planner's pure subjective rate of time discount. The terminal requirement on the capital-labor ratio, i.e., $k(T) \geq k_T$, deserves comment. Although it is very natural to insist that the planner is bound by his inheritance $k(0) = k_0$, it is somewhat difficult to make an argument for what form the constraint on terminal factors should be. If T is finite, $k(T)$ cannot in general be left free because there is the post-planning period to worry about. It is this sort of argument that makes the choice of the infinite planning horizon, $T = \infty$, attractive. For the time being, let it suffice to say that in practice the planning horizon is finite, $T < \infty$. The terminal constraint, $k(T) \geq k_T$, is an inequality rather than an equality, since cases in which the maximand must be depressed in order not to overfulfill the terminal requirement should be considered objectionable.

The above planning problem is solved by employing the Maximum Principle of Pontryagin.[8] Define the Hamiltonian $H(k, s, t, q)$ by

$$He^{\delta t} \equiv (1 - s)f(k) + q[sf(k) - \lambda k],$$

where $q(t)e^{-\delta t}$ is a Hamiltonian multiplier. Applying Theorem 3* (page 63), the Maximum Principle yields that if a program[9] $\{k(t), s(t): t \in [0, T]\}$ is optimal, then there exists a continuous function $q(t)$ such that

$$\begin{cases} \overset{\circ}{k} = sf(k) - \lambda k, \\ \overset{\circ}{q} = (\delta + \lambda)q - [1 - s + qs]f'(k), \\ s \text{ maximizes } [1 - s + qs] \text{ subject to } s \in [0, 1]. \end{cases}$$

Instantaneous maximization of H with respect to $s \in [0, 1]$ is a simple linear programming problem which yields

$$\max_{s\in[0,1]} [1 - s + qs] = \max(1, q) \equiv \gamma(q) .$$

That is, $s(\cdot)$ is an upper semicontinuous correspondence which is given by

$$s(q) \begin{cases} = 1, & \text{when } q > 1, \\ \in [0, 1], & \text{when } q = 1, \\ = 0, & \text{when } q < 1. \end{cases}$$

The measurability requirement may be of the form that $s(t)$ be (say) a piece-wise continuous function of t. $q(t)$ has the interpretation of the social demand price of investment at time t (in terms of consumption foregone at time t). Thus the differential equation $\overset{\circ}{q} = (\delta + \lambda)q - \gamma f'(k)$ has the following interpretation. In an economy in which capital rental is rewarded by its marginal value product, the price of a unit of capital must change so as to reward the <u>rentier</u> for "waiting" less the value of net rentals received, $\gamma f'(k) - \lambda q$.

In addition to satisfying the Maximum Principle, an optimal trajectory must meet certain transversality conditions.[10] The left-hand endpoint (or transversality) condition, $k(0) = k_0$, is trivial but the right-hand transversality condition is nontrivial. For an "interior" maximum with respect to k(T) we know that it is required that at T demand price discounted back to time 0 (the Hamiltonian multiplier) be orthogonal to all capital stock (or state) variables in the feasible manifold, i.e.,

$$q(T)e^{-\delta T}k(T) = 0$$

But since feasibility implies that $0 < k(T) < \infty$, the "interior" endpoint condition is that $q(T) = 0$. Allowing for a corner endpoint maximum, the full transversality condition is

$$q(T)e^{-\delta T}[k(T) - k_T] = 0 \quad \text{and} \quad q(T)e^{-\delta T} \geq 0 .$$

That is, it is necessary for a maximum that either the terminal value of the social demand price of capital be zero or that the terminal requirement be met with equality. Restated once more: the transversality condition expresses the planner's desire not to have capital "left over" (where capital is valued at demand prices, $q(T)k(T)$) beyond terminal requirements, $q(T)k_T$.

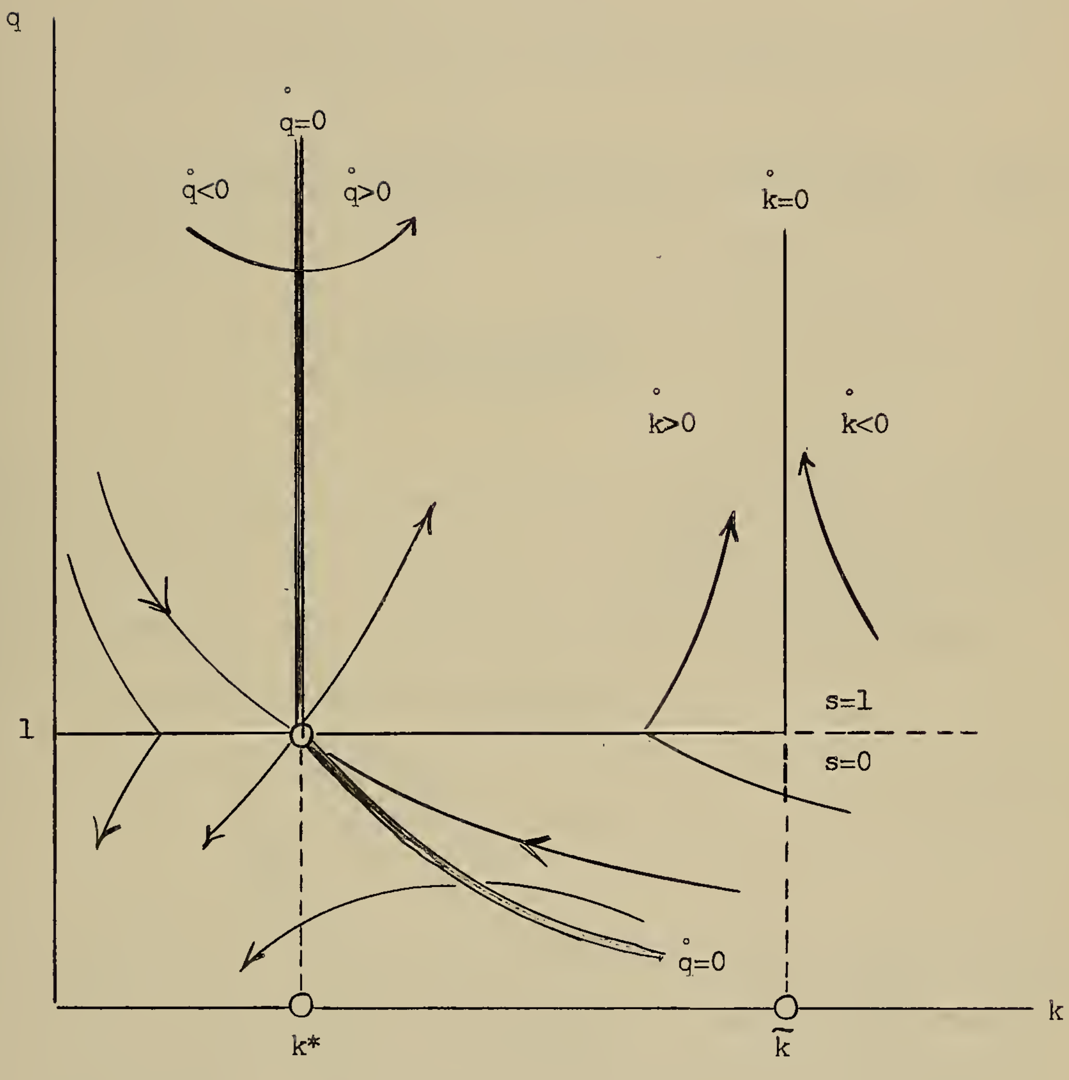
Next it is required to study the dynamical system that is implied by Pontryagin's necessary conditions. The dynamical system does not reduce to a system of differential equations. This is because instantaneous

maximization implies that s is an upper semicontinuous correspondence in q-- rather than a function of q. Thus we must treat a law of motion for accumulation of capital that is not in the form of a differential equation. Our dynamical system includes the law "$\dot{k} \in \Omega(q,k)$," where $\Omega(q, k)$ is a set. In the terminology of Professor Yorke's lectures, we are treating a "flow without uniqueness." In preparing this lecture, I did not take into account the work of Professor Yorke and was thus forced to adapt techniques particular to my own special problem. Perhaps, then, my treatment will serve as a particular example for the more general problems treated by the mathematicians in that field.

Now, "nonuniqueness of flow" is not a new subject for economics. In Lecture IV, I will show how the same problem appears in descriptive models of heterogeneous capital accumulation. In that and other contexts economists refer to the problem as that of "nonuniqueness of momentary equilibrium."

In the one-sector model, with the linear criterion functional, it is quite clear why momentary equilibrium is nonunique. In consumption-investment space (C, Z), the production possibility set is by definition a right isosceles triangle. The absolute value of the slope of the production possibility frontier is unity. But the maximand can be written as C + qZ. If $q > 1$, instantaneous maximization implies specialization to investment, $s = 1$. If $q < 1$, instantaneous maximization implies specialization to consumption, $s = 0$. But if $q = 1$, all points along the production possibility frontier are consistent with instantaneous maximization, $0 \leq s \leq 1$.

Figure 2 is the phase portrait representing the dynamical system



q
$\mathring{q}=0$
$\mathring{q}<0$
$\mathring{q}>0$
$\mathring{k}=0$
$\mathring{k}>0$
$\mathring{k}<0$
s=1
1
s=0
$\mathring{q}=0$
k
k*
$\tilde{k}$

Figure 2

$$
(DS) \begin{cases} \overset{\circ}{k} = sf(k) - \lambda k, \\ \overset{\circ}{q} = (\delta + \lambda)q - \max(1, q)f'(k), \\ s = 1 \text{ for } q > 1,\ s = 0 \text{ for } q < 1,\ s \in [0, 1] \text{ for } q = 1 . \end{cases}
$$

It is required to study the stationary solutions to this system. Notice that $\overset{\circ}{q} = 0$ if and only if

$$
q = \frac{\max(1, q)f'(k)}{\delta + \lambda} .
$$

Hence for $q \geq 1$, $\overset{\circ}{q} = 0$ if and only if $k = k^*$, where as shown in Figure 1, k^* is the unique solution to $f'(k) = \delta + \lambda$. Since $f(\cdot)$ is strictly concave, in the "usual" case with $\delta > 0$, we have, as is shown in Figure 1, that $0 < k^* < \hat{k} < \tilde{k}$. Differentiation yields that

$$
\left.\frac{dq}{dk}\right|_{\overset{\circ}{q}=0} = \frac{f''(k)}{\delta + \lambda} < 0, \text{ for } q < 1 .
$$

For $q < 1$, there are no stationaries to the capital accumulation equation. For $q > 1$, $\overset{\circ}{k} = 0$ if and only if k is equal to the maximum sustainable capital-labor ratio $\tilde{k}$. Of course for $k > \tilde{k}$, there are no stationary solutions to the capital accumulation law. However for $0 < k \leq \tilde{k}$, and $q = 1$, $\overset{\circ}{k} = 0$ if and only if $s = \lambda k/f(k)$.

Now to study the dynamical system (DS) in greater detail. Make the assignment $s(q) = 1$ in the RHS of the first two lines of (DS). Clearly this

new dynamical system has RHS which are continuously differentiable functions of the arguments k and q on the domain $\{k > 0, q \geq 1\}$. Further by assigning $s(q) = 0$, the RHS are seen to be continuously differentiable functions on the domain $\{k > 0, q \leq 1\}$. Thus, when the controller s is appropriately assigned, (DS) is a trivially Lipschitzian system of differential equations over the two open sets $\{k > 0, q > 1\}$ and $\{k > 0, q < 1\}$.

When $q = 1$, the value of $s \in [0, 1]$ is indeterminate. But if $q = 1$ and $k \gtrless k^*$, $\dot{q} \gtrless 0$. Thus for $k \neq k^*$, the indeterminacy of s at $q = 1$ "only lasts for an instant" and consequently the trajectory $\{k(t)\}$ is independent of what assignment of $s \in [0, 1]$ is made at points $(k \neq k^*, q = 1)$. Therefore, by classic theorems in the theory of ordinary differential equations,[11] we know that for the dynamical system (DS) that specification of the parameters $(k(t_0), q(t_0))$ uniquely determines the entire dynamical behavior of the system for those trajectories that do not pass through $(k = k^*, q = 1)$. In fact, for trajectories not passing through $(k = k^*, q = 1)$, "solutions" $(k(t_1), q(t_1))$ to the system (DS) vary continuously with respect to initial conditions $(k(t_0), q(t_0))$.

For the system (DS), when $k = k^*$ and $q = 1$ not only is the value of the controller $s \in [0, 1]$ not uniquely determined, but also this nonuniqueness is essential in the sense that the trajectory $\{k(t), q(t): t \in [0, \infty]\}$ will depend upon the value of s that is assigned at $(k^*, 1)$. If at $(k^*, 1)$, $s = s^* \equiv f(k^*)/\lambda k^*$, then $\dot{q} = 0$ and $\dot{k} = 0$. If $s < s^*$, then $\dot{q} < 0$ and $\dot{k} < 0$; if $s > s^*$ then $\dot{q} > 0$ and $\dot{k} > 0$. It should be remarked that if we add to (DS) the condition that $s = s^*$ when $q = 1$, then the new system is now a system of

differential equations with a discontinuity in the RHS at $q = 1$. For this system of differential equations, $(k^*, 1)$ is the unique equilibrium (or rest point). Further since the stable arms are unique, $(k^*, 1)$ is a saddlepoint in the (k, q) phase space.

Description of trajectories satisfying Pontryagin's necessary conditions. For T sufficiently large, the Pontryagin program (i.e., the feasible program satisfying (DS) and the transversality condition $k(0) = k_0$, $q(T)e^{-\delta T} \geq 0$, and $e^{-\delta T}q(T)[k(T) - k_T] = 0$) is found as follows. (1) Assign $q(0)$ such that $(k(0) = k_0, q(0))$ lies on the unique stable arm of the saddlepoint $(k^*, 1)$. It is clear that the stable arm covers the open half line $0 < k < \infty$. (2) Assign $q(T) \geq 0$ such that $(k(T) = k_T, q(T))$ lies on the unique backward solution to $(k^*, 1)$. If this is inconsistent with preservation of the nonnegativity of $q(T)$ then set $q(T) = 0$ and choose $k(T) > k_T$ so that $(q(T) = 0, k(T) > 0)$ is on the unique backward solution to $(k^*, 1)$. (3) Designate the Pontryagin value of the capital-labor ratio by k^0 and similarly for the other variables: q^0, z^0, etc. . . . Denote the time at which the forward solution from $(k^0(0), q^0(0))$ reaches $(k^*, 1)$ by t^*. Denote the time at which the backward solution from $(k^0(T), q^0(T))$ reaches $(k^*, 1)$ by t^{**}. Then the Pontryagin savings ratio $s^0 = f(k^*)/\lambda k^*$ for $t \in [t^*, t^{**}]$.

Figure 3 shows an example of a program satisfying Pontryagin's necessary conditions. Important assumptions are implicit in the construction of Figure 3. It is assumed that it is feasible for the economy with initial endowment $k(0) = k_0$ to achieve the target k_T in the specified time T. Even stronger, Figure 3 assumes that in fact $T > t^{**} > t^* > 0$. If it is feasible

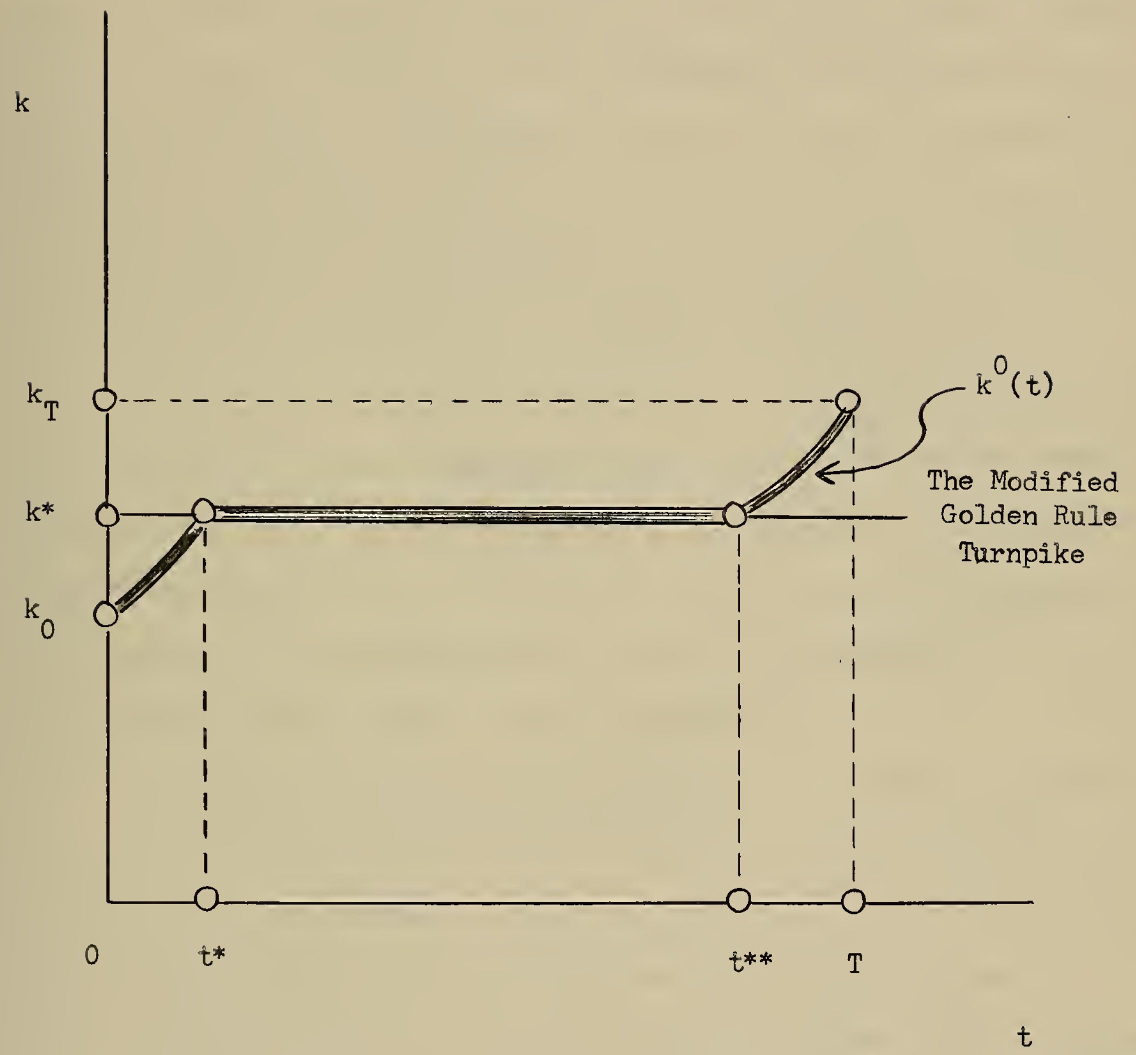


Figure 3

to achieve the target during the planning period but it is not true that $T > t^{**} > t^* > 0$, then the Pontryagin path is the appropriate envelope of a forward trajectory from $(k_0, q(0))$ to $(k^*, 1)$ and the backward trajectory from $(k(T), q(T))$ to $(k^*, 1)$. In the degenerate case in which only one feasible path exists, the Pontryagin path is, of course, a program that is specialized to investment (i.e., $c(t) = 0$ for $t \in [0,T]$). Since we do not permit the demand price of investment to be negative, if no trajectory is found with $k(T) = k_T$ and $q(T) \geq 0$ then $k^0(T) > k_T$ while $q^0(T) = 0$.

If we call k^* the Modified Golden Rule capital-labor ratio, then we have proved the following Turnpike Theorem: For the one-sector planning problem above, following the Pontryagin program requires the planner to adopt the Modified Golden Rule of Accumulation for all but a finite amount of time. As the length of the planning period increases the fraction of time with a capital-labor ratio different from k^* approaches zero.

Optimality of the Pontryagin Program. It remains to show that for our problem Pontryagin's necessary conditions are also sufficient, that such programs are indeed optimal. Let $\{c^0(t), z^0(t), k^0(t), q^0(t), \ldots\}$ be a program satisfying (DS) and the transversality conditions. Let $\{c(t), z(t), k(t), q(t), \ldots\}$ be any feasible program. It is required to show

$$\int_0^T (c^0 - c)e^{-\delta t}\, dt \geq 0 \;.$$

The LHS can be rewritten as

$$\int_0^T e^{-\delta t}\, dt \Big\{(c^0 - c) + \gamma^0[(f(k^0) - z^0 - c^0) - (f(k) - z - c)]$$

$$+ q^0[(z^0 - \lambda k^0 - \dot{k}^0) - (z - \lambda k - \dot{k})]\Big\},$$

which reduces to

$$\int_0^T e^{-\delta t}\, dt \Big\{(1 - \gamma^0)(c^0 - c) + (q^0 - \gamma^0)(z^0 - z)$$

$$+ \gamma^0[f(k^0) - f(k)] + q^0[\lambda(k - k^0) + (\dot{k} - \dot{k}^0)]\Big\}.$$

Notice that

$$(1 - \gamma^0)(c^0 - c) \geq 0$$

and

$$(q^0 - \gamma^0)(z^0 - z) \geq 0 .$$

Therefore $\int_0^T (c^0 - c)e^{-\delta t} dt$ is not less than the following expression

$$\int_0^T e^{-\delta t} dt \Big\{\gamma^0[f(k^0) - f(k)] + q^0[\lambda(k - k^0) + (\dot{k} - \dot{k}^0)]\Big\} .$$

But since $f(\circ)$ is a concave function, the above expression is not smaller than

$$\int_0^T e^{-\delta t}\, dt\{\gamma^0[(k^0 - k)f'(k^0)] + q^0[\lambda(k - k^0) + (\dot{k} - \dot{k}^0)]\} .$$

By collecting terms the expression above yields

$$\int_0^T q^0 e^{-\delta t}(\dot{k} - \dot{k}^0)dt + \int_0^T e^{-\delta t}\, dt(k^0 - k)\{\gamma^0 f'(k^0) - q^0\lambda\} .$$

Integrating the first term above by parts yields

$$q^0(T)e^{-\delta T}\{k(T)-k^0(T)\} - q_0^0\{k(0)-k^0(0)\} - \int_0^T (k-k^0)(\dot{q}^0-\delta q^0)e^{-\delta t}dt .$$

The transversality condition says that the first term above is nonnegative. Since every feasible path must satisfy the given initial condition k_0, the second term above is identically zero. Hence

$$\int_0^T q^0 e^{-\delta t}(\dot{k} - \dot{k}^0)dt \geq -\int_0^T (k - k^0)(\dot{q}^0 - \delta q^0)e^{-\delta t}dt .$$

Hence $\int_0^T (c^0 - c)e^{-\delta t}dt$ is not smaller than

$$\int_0^T e^{-\delta t}dt[(k^0 - k)\{\gamma^0 f'(k^0) - \lambda q^0\} + (k^0 - k)(\dot{q}^0 - \delta q^0)]$$

$$= \int_0^T e^{-\delta t}dt(k^0 - k)\{\dot{q}^0 - (\delta + \lambda)q^0 + \gamma^0 f'(k^0)\}$$

which by (DS) is identically zero. Hence the optimality requirement is established. In fact, if $k \neq k^0$ on some interval then the above inequality is strict.

Footnotes for Lecture I

1. Frank P. Ramsey, "A Mathematical Theory of Saving," Economic Journal, Vol. 38, 1928, pp. 543-59.
2. Two such important papers are: P. A. Samuelson and R. M. Solow, "A Complete Capital Model Involving Heterogeneous Capital Goods," Quarterly Journal of Economics, Vol. 70, November 1956, pp. 537-62, and T. C. Koopmans, "On the Concept of Optimal Economic Growth" in Semaine d'Etude sur Le Rôle de l'Analyse Econométrique dans la Formulation de Plans de Dévelopement, October 1963, Pontifical Academy of Sciences, Vatican City, 1965, pp. 225-87.
3. For the seminal paper in descriptive one-sector growth theory, see R. M. Solow, "A Contribution to the Theory of Economic Growth," Quarterly Journal of Economics, Vol. 70, February 1956, pp. 65-94.
4. In this lecture, I will omit explicit indications of time dependence when such omission does not lead to confusion.
5. For references to the GR literature and its history, see E. S. Phelps, "Second Essay on the Golden Rule of Accumulation," American Economic Review, Vol. 55, No. 4, September 1965, pp. 793-814.
6. See Phelps, op. cit.
7. Or consumption per citizen if workers are a given constant fraction of the total population.
8. L. S. Pontryagin et al, The Mathematical Theory of Optimal Processes, New York and London: Interscience Publishers, 1962. See especially Chapter I, pp. 1-74.

9. Pontryagin *et al*, *op. cit.*, pp. 17-74. The correspondence between my notation and that of Pontryagin *et al.* follows:

Pontryagin et al.	This lecture
x	k
u	s
t	t
t_0	0
t_1	T
U	$[0, 1]$
$f^0(x, u, t)$	$(1-s)f(k)e^{-\delta t}$
$f(x, u, t)$	$sf(k) - \lambda k$
ψ_0	1
$\psi(t)$	$q(t)e^{-\delta t}$

The planning problem treated in this lecture is a special case of the problem treated in my "Optimal Programs of Capital Accumulation for an Economy in Which There is Exogenous Technical Change," in K. Shell (ed.), *Essays on the Theory of Optimal Economic Growth*, Cambridge, Mass., and London: M.I.T. Press, 1967.

10. Pontryagin *et al*, *op. cit.*, pp. 45-57.

11. Cf., e.g., L. S. Pontryagin, *Ordinary Differential Equations*, Reading, Mass.: Addison-Wesley, 1962, especially pp. 159-67 and pp. 192-99.

LECTURE II

In the previous lecture, I derived consumption-optimal trajectories for the one-sector model with linear objective functional and constant time discounting. Optimal trajectories of the savings ratio $\{s^0(t): t \in [0, T]\}$ were found to be piecewise constant, where $s^0(t)$ is equal to 0 <u>or</u> 1 <u>or</u> $s^* = f(k^*)/\lambda k^*$. Economists have referred to such trajectories as "bang-bang" solutions, although this is not precisely what is meant by a "bang-bang" solution in the usual control theory literature. (In engineering control theory, an optimal control is said to be "bang-bang" if the optimal controller only takes on values on the boundary of the control set. In our case that would imply that $s^0(t)$ is equal to 0 <u>or</u> 1). Furthermore, it was shown that optimal trajectories possess an <u>exact</u> turnpike property; i.e., for sufficiently large planning periods the optimal capital-labor ratio is <u>exactly</u> equal to the Modified Golden Rule capital-labor ratio for "most" of the planning period. In this lecture I would like to indicate how the exactness of the turnpike property and the essential discontinuity of the optimal controller $s^0(t)$, $t \in [0, T]$, depend upon two important linearities in the problem of Lecture I--viz.: (1) the linearity of the production possibility frontier in consumption-investment space (c, z), and (2) the linearity of the Hamiltonian H (or net national product at socially imputed demand prices) in (c, z) space.

Our first example is the planning problem for the two-sector model with linear objective functional. If efficient capital intensities differ in

the two sectors, then the optimal savings ratio $s^0(t)$ is continuous in time. Furthermore, although the optimal trajectory possesses a Modified Golden Rule turnpike property, it is not an exact turnpike property.

Consider the two-sector model of economic growth which was introduced by Meade[1] and further articulated by Uzawa.[2] The model economy consists of an investment-goods sector and a consumption-goods sector, labelled 1 and 2, respectively. In both sectors, production is subject to constant-returns-to-scale, and marginal rates of substitution are positive. There are no external economies (diseconomies) and no joint products.

The quantity of the consumption goods, $Y_2(t)$, produced at time t depends upon the respective allocations at time t, $K_2(t)$ and $L_2(t)$, of capital and labor to the consumption sector,

$$Y_2(t) = F_2[K_2(t), L_2(t)] . \tag{II.1}$$

Similarly, production of the investment goods at time t, $Y_1(t)$ is dependent upon the allocation of factors to the investment sector

$$Y_1(t) = F_1[K_1(t), L_1(t)] . \tag{II.2}$$

$F_1[\cdot]$ and $F_2[\cdot]$ are neoclassical production functions, positively homogeneous of degree one in their respective arguments. Thus

$$F_j[\theta K_j, \theta L_j] = \theta F_j[K_j, L_j] \qquad \text{for } j = 1, 2,$$

where K_j, $L_j \geq 0$ and $\theta > 0$. Labor and capital are freely shiftable between the two sectors.[3] For an allocation of resources to be feasible at time t $K_1(t) + K_2(t) \leq K(t)$ and $L_1(t) + L_2(t) \leq L(t)$, with $K_1(t)$, $K_2(t)$, $L_1(t)$, $L_2(t) \geq 0$ where $K(t) > 0$ and $L(t) > 0$ are the stocks of available capital and labor at time t.

If the capital stock is subject to evaporative decay at the constant rate $\mu > 0$, then growth of the capital stock is specified by the differential equation

$$\text{(II.3)} \qquad \mathring{K}(t) = Y_1(t) - \mu K(t) .$$

If labor is inelastically supplied and is growing at the exogenously given exponential rate n, then $\mathring{L} = nL$.

The problem is to characterize the program of capital accumulation that maximizes the integral of discounted per capita consumption over a specified planning period subject to the feasibility conditions (above) and satisfying the given initial conditions and satisfying the appropriate terminal targets.

Formally the problem is <u>to maximize</u>

$$\int_0^T \frac{C(t)}{L(t)} e^{-\delta t} dt$$

where δ is the social rate of (constant) time discount and $\infty > T > 0$ is the

length of the planning period. The maximand is constrained by the system (II.1) - (II.3) and by the given initial conditions $K(0) = K_0$, $L(0) = L_0$, and subject to the requirement that the terminal capital-labor ratio be at least as great as some specified target, $k(T) \geq k_T$.[4]

The Two-Sector Model: Preliminaries. In order to facilitate the exposition, certain techniques developed by Uzawa, and Oniki and Uzawa must be presented. First define the per capita quantities:

$$y_j(t) = \frac{Y_j(t)}{L(t)} \;;\quad k_j(t) = \frac{K_j(t)}{L_j(t)} \;;\quad \ell_j(t) = \frac{L_j(t)}{L(t)} \;;\quad f_j(k_j) = F_j(k_j, 1) ,$$

for $j = 1, 2$; $k(t) = \dfrac{K(t)}{L(t)}$.

Assume that $f_j(k_j)$ is twice continuously differentiable and

$$f_j(k_j) > 0 \quad f_j'(k_j) > 0, \quad f_j''(k_j) < 0 \quad \text{for } 0 < k_j < \infty$$

$$\lim_{k_j \downarrow 0} f_j(k_j) = 0 \qquad \lim_{k_j \uparrow \infty} f_j(k_j) = \infty$$

$$\lim_{k_j \downarrow 0} f_j'(k_j) = \infty \qquad \lim_{k_j \uparrow \infty} f_j'(k_j) = 0$$

Because positive marginal products are assumed, optimality requires that $K_1 + K_2 = K$ and $L_1 + L_2 = L$. If ω is an arbitrarily given wage-rentals ratio, then efficient capital-labor ratios can be found by solving for k_j

(II.4) $$\omega = \frac{f_j(k_j)}{f'_j(k_j)} - k_j \qquad \text{for } j = 1, 2,$$

and therefore

$$\frac{dk_j}{d\omega} = \frac{-[f'_j(k_j)]^2}{f_j(k_j)f''_j(k_j)} > 0 \ .$$

Thus the efficient capital-labor ratio k_j is a uniquely determined, increasing function of the wage-rentals ratio ω.

Define $p(\omega)$ the supply price of a unit of investment goods

$$p(\omega) = \frac{f'_2(k_2(\omega))}{f'_1(k_1(\omega))} \ ,$$

where a unit of consumption goods is the numeraire.

Logarithmic differentiation yields

(II.5) $$\frac{1}{p}\frac{\partial p}{\partial \omega} = \frac{1}{k_1(\omega) + \omega} - \frac{1}{k_2(\omega) + \omega} \gtrless 0 \text{ as } k_2 \gtrless k_1 \ .$$

Full employment implies that

$$k_1\ell_1 + k_2\ell_2 = k \ ,$$

and

$$\ell_1 + \ell_2 = 1 \ ,$$

where $k_1, k_2, \ell_1, \ell_2 \geq 0$. Given k, the full employment conditions above

define the range of p and ω. This is illustrated in Figure II.1 for the case $k_2(\omega) > k_1(\omega)$. In general define the critical wage-rentals ratios by

$$\omega_{min}(k) = \min\{\omega_2(k), \omega_1(k)\} ,$$

$$\omega_{max}(k) = \max\{\omega_2(k), \omega_1(k)\} ,$$

and the critical supply prices by

$$p_{min}(k) = p_2(k) ,$$

$$p_{max}(k) = p_1(k) ,$$

where $k_j(\omega_j) = k$ and $p_j(k) = p(\omega_j(k))$ for $j = 1, 2$.

Solving from above yields

$$y_1 = \frac{k_2 - k}{k_2 - k_1} f_1(k_1) ,$$

(II.6)

$$y_2 = \frac{k - k_1}{k_2 - k_1} f_2(k_2) .$$

Thus given k, ω, we have uniquely determined the efficient allocation of factors between the two sectors; the level of production in both sectors; and the supply price of a unit of the investment good in terms of a unit of the consumption good.[5] Partial differentiation of (II.6) yields

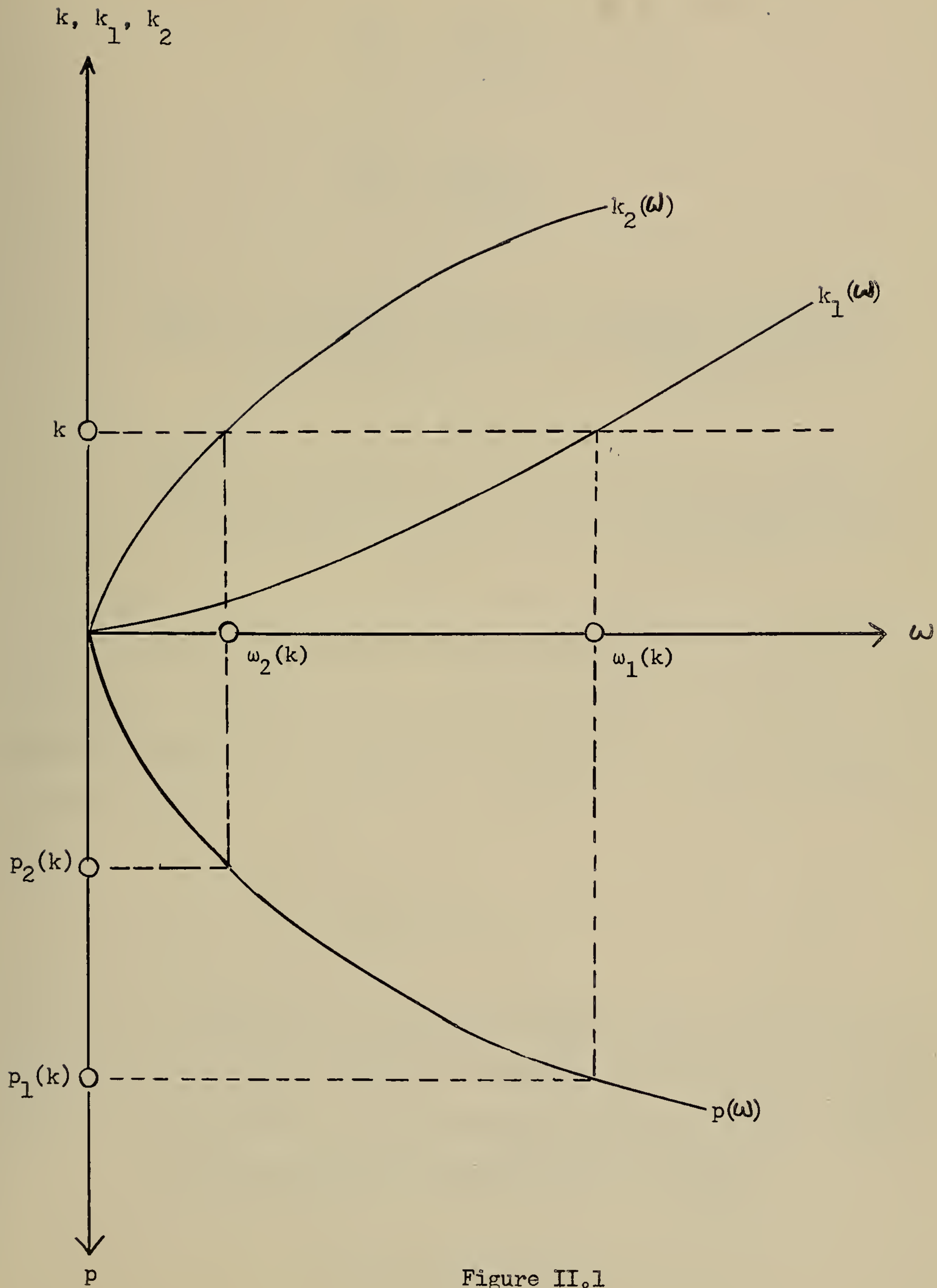


Figure II.1

$$\frac{\partial y_1}{\partial k} = \frac{-f_1(k_1)}{k_2 - k_1} ,$$

$$\frac{\partial y_2}{\partial k} = \frac{f_2(k_2)}{k_2 - k_1} ,$$

$$\frac{\partial y_1}{\partial \omega} = \left(\frac{k_2 - k}{(k_2-k_1)^2}\right)\left(\frac{f_1'^2}{f_1 f_1''}\left[-k_2 f_1' - (f_1 - k_1 f_1')\right]\right) + \left(\frac{k_1 - k}{(k_2-k_1)^2}\right)\left(\frac{f_2'^2 f_1}{f_2 f_2''}\right) ,$$

$$\frac{\partial y_2}{\partial \omega} = \left(\frac{k - k_1}{(k_2-k_1)^2}\right)\left(\frac{f_2'^2}{f_2 f_2''}\left[k_1 f_2' + (f_2 - k_2 f_2')\right]\right) + \left(\frac{k_2 - k}{(k_2-k_1)^2}\right)\left(\frac{f_1'^2 f_2}{f_1 f_1''}\right) .$$

The supply value of gross national product per worker, y, is defined by $y = y_2 + py_1$. It is useful to define s, the fraction of implicit gross national product assigned to investment, $s = py_1/y$. Rewriting, the maximand becomes

$$\int_0^T y_2 e^{-\delta t} dt .$$

Without loss in generality, the central planning board can consider $s(t)$ to be the control variable chosen from among all (say) piecewise continuous functions defined upon $t \in [0, T]$ such that $s(t) \in [0, 1]$ for $t \in [0, T]$.

The problem reduces to choosing $s(t)$ to maximize

$$\int_0^T (1 - s(t))y(t)e^{-\delta t} dt ,$$

subject to the constraints:

$$\dot{k}(t) = \frac{s(t)y(t)}{p(t)} - \lambda k \qquad \text{where } \lambda = \mu + n ,$$

$0 \leq s(t) \leq 1$ for $0 \leq t \leq T$ and $s(t)$ a piecewise continuous function of t,

$k(0) = k_0$ and $k(T) \geq k_T$.

The above problem is soluble as an application of Pontryagin's Maximum Principle. First form the Hamiltonian expression $H(k, q, s, t) = e^{-\delta t}\{(1-s)y + q[\frac{sy}{p} - \lambda k]\}$, where q(t) is interpreted as the social demand price of a unit of the investment good in terms of a unit of the consumption good. It is necessary for optimality that s(t) be chosen in $[0, 1]$ to maximize the socially imputed value of gross national product at time t, $(1-s)y + q(sy/p)$. Thus it is necessary for optimality that:

$s=1,\ k_1=k,\ \ell_1=1$; <u>or</u>

$s=0,\ k_2=k,\ \ell_2=1$; <u>or</u>

$$\frac{\partial(1-s)y}{\partial s} + q\,\frac{\partial(\frac{sy}{p})}{\partial s} = 0 .$$

The equation immediately above can be rewritten as

$$\frac{\partial y_1}{\partial \omega}\frac{\partial \omega}{\partial s} = - q \frac{\partial y_2}{\partial \omega}\frac{\partial \omega}{\partial s} .$$

But we have derived that

$$\left(\frac{\frac{\partial y_2}{\partial \omega}}{\frac{\partial y_1}{\partial \omega}}\right) = -p .$$

Hence if the instantaneous maximum is interior, then, given q, s is chosen such that $p(k, s) = q$. Interior maxima are characterized by the equality of demand and supply prices.

Pontryagin's second necessary condition is that the social demand price of investment change through time in a manner reflecting the planning board's perfect foresight of the imputed marginal value product of capital,

$$\text{(II.7)} \qquad \overset{\circ}{q} = (\delta+\lambda)q - \left\{\frac{\partial(1-s)y}{\partial k} + q\frac{\partial(\frac{sy}{p})}{\partial k}\right\} .$$

But

$$\frac{\partial(\frac{sy}{p})}{\partial k} = \frac{\partial y_1}{\partial k} + \frac{\partial y_1}{\partial \omega}\frac{\partial \omega}{\partial k} = -f_1'\left(\frac{k_1+\omega}{k_2-k_1}\right) + \frac{\partial y_1}{\partial \omega}\frac{\partial \omega}{\partial k} ,$$

and

$$\frac{\partial(1-s)y}{\partial k} = \frac{\partial y_2}{\partial k} + \frac{\partial y_2}{\partial \omega}\frac{\partial \omega}{\partial k} = p\left\{ f_1'\left(\frac{k_2+\omega}{k_2-k_1}\right) - \frac{\partial y_1}{\partial \omega}\frac{\partial \omega}{\partial k}\right\} .$$

Therefore if $p(k, s) = q$, then differential equation (II.7) reduces to

(II.7.N) $$\dot{q} = \left\{\delta+\lambda-f_1'(k_1)\right\}q \quad .$$

Indeed if $p(k, s) \leq q$, then (II.7) reduces to (II.7.N). But if $p(k, s) > q$, then (II.7) reduces to

(II.7.S_2) $$\dot{q} = (\delta+\lambda)q - f_2'(k) \quad .$$

The Case where Production of the Consumption Good is More Capital Intensive than Production of the Investment Good

It is convenient to treat the general problem posed above by separate cases depending upon certain attributes of the production functions (II.1) and (II.2). Now I treat the case where $k_2(\omega) > k_1(\omega)$, for $\omega > 0$. The (k, q)-phase plane is divided into three mutually exclusive regions:

$$S_1 = \left\{(k, q) : q > p_{max}(k)\right\} ,$$

$$S_2 = \left\{(k, q) : q < p_{min}(k)\right\} ,$$

$$N = \left\{(k, q) : p_{min}(k) \leq q \leq p_{max}(k)\right\} .$$

In region S_1, maximization of the demand value of gross national product implies specialization to the production of the investment good. Therefore,

$$\text{(II.8.}S_1\text{)}\qquad \begin{aligned} \mathring{k} &= f_1(k) - \lambda k, \text{ in } S_1, \text{ and} \\ \mathring{q} &= \left\{(\delta+\lambda) - f_1'(k)\right\}q, \text{ in } S_1 . \end{aligned}$$

And similarly,

$$\text{(II.8.}S_2\text{)}\qquad \begin{aligned} \mathring{k} &= -\lambda k, \text{ in } S_2, \text{ and} \\ \mathring{q} &= (\delta+\lambda)q - f_2'(k), \text{ in } S_2 . \end{aligned}$$

In region N (for nonspecialization),

$$\text{(II.8.N)}\qquad \begin{cases} \mathring{k} = f_1(k_1(\omega))\,\dfrac{k_2(\omega)-k}{k_2(\omega)-k_1(\omega)} - \lambda k & \text{in N, and} \\ \dot{q} = \left\{(\delta+\lambda) - f_1'(k_1(\omega))\right\}q & \text{in N,} \\ q(t) = p(\omega) & \text{in N .} \end{cases}$$

Since, by assumption $k_2(\omega) > k_1(\omega)$, p is a strictly increasing function of ω. Therefore specification of q uniquely determines ω, which in

turn uniquely determines $k_1(\omega)$ and $k_2(\omega)$. Thus, the RHS of (II.8.N) is uniquely determined by specification of $(k, q) \in N$.

For $(k, q) \in S_1$, $\overset{\circ}{q} = 0$ if and only if $f_1'(k) = \delta+\lambda$. Call the unique solution to this equation k_1^*. Define $q^* = p_{max}(k_1^*)$. Since p_{max} is an increasing function of k, q^* is uniquely defined. Further define ω^* as the unique solution to $q^* = p(\omega)$. Since $k_1^* = k_1(\omega^*)$, we have that for $(k, q) \in N$, $\dot{q} = 0$ if and only if $q(t) = q^*$.

For $(k, q) \in S_2$, $\overset{\circ}{q} = 0$ if and only if $q = f_2'(k_2)/\delta+\lambda$. That is,

$$\left(\frac{\partial q}{\partial k}\right)_{\overset{\circ}{q}=0} = \frac{f_2''(k)}{\delta+\lambda} < 0, \text{ for } (k, q) \in S_2 .$$

Next it is required to describe the set of points that yield stationary solutions to the capital accumulation equation in (II.8). For $(k, q) \in S_1$, $\dot{k} = 0$ if and only if $f_1(k) = \lambda k$. Concavity of $f_1(\cdot)$ along with the "Inada endpoint assumption" ensures that $\tilde{k}$, the maximum sustainable capital-labor ratio is uniquely defined.

For $(k, q) \in N$ <u>and</u> $k > \tilde{k}$, there are no stationary solutions to the capital accumulation equation. However, for $(k, q) \in N$ <u>and</u> $k < \tilde{k}$, stationary solutions to the capital accumulation equation are such that $p_{min}(k) < q < p_{max}(k)$. Of course, for $(k, q) \in S_2$ there are no nontrivial stationary solutions to the capital accumulation equation.

The system of differential equations (II.8) is autonomous and thus can be characterized by the two-dimensional phase diagram of Figure II.2. The intersection of the locus $\overset{\circ}{q} = 0$ with the locus $\overset{\circ}{k} = 0$ is shown to be

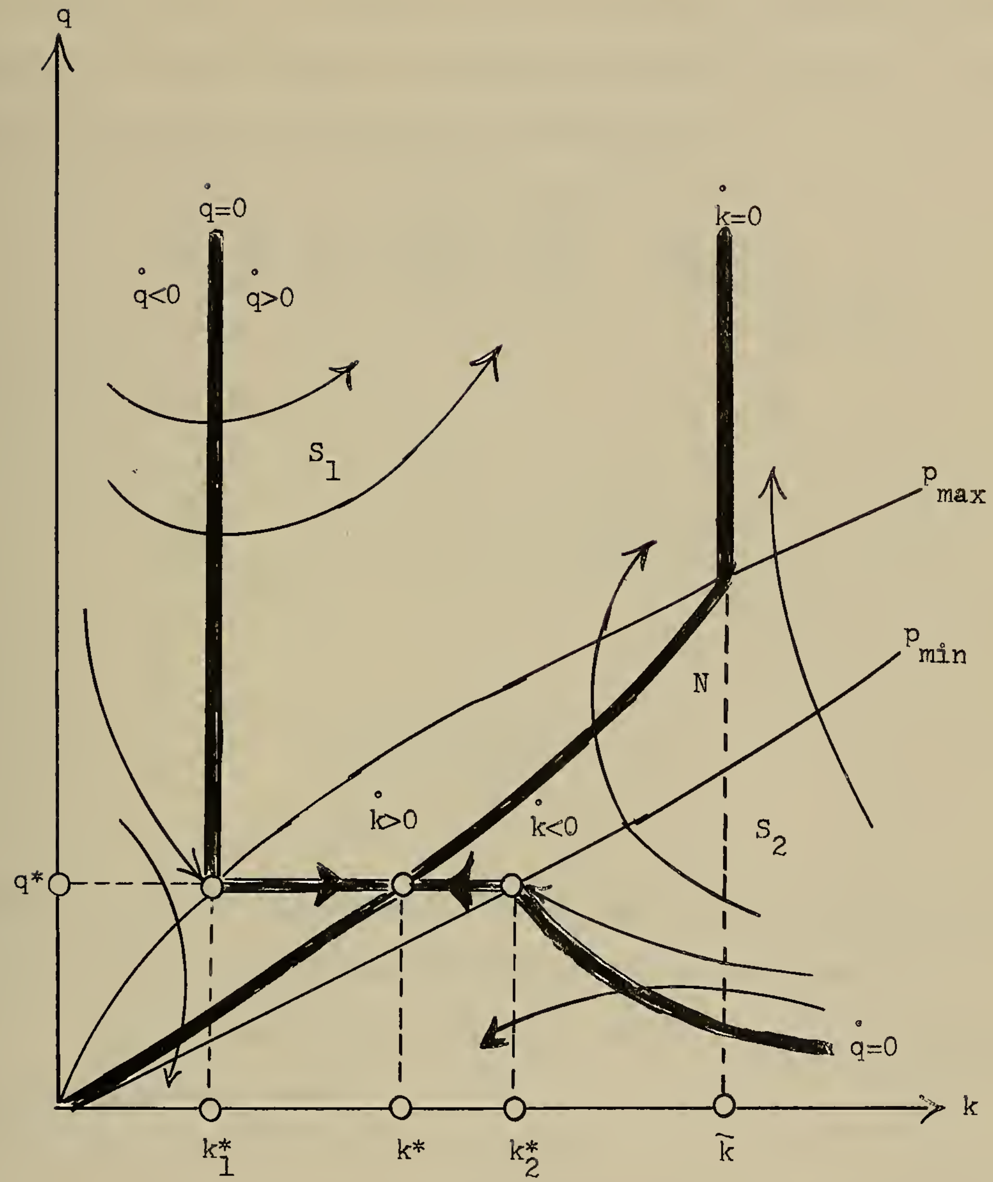


Figure II.2

the point (k*, q*). To verify that (k*, q*) is a saddlepoint for the system, consider the linear Taylor approximation (II.8.N) evaluated at (k*, q*). The roots to the relevant characteristic equation are

$$\frac{1}{2}\left[\frac{\partial \dot{k}}{\partial k} + \frac{\partial \dot{q}}{\partial q} \pm \sqrt{\left(\frac{\partial \dot{k}}{\partial k} + \frac{\partial \dot{q}}{\partial q}\right)^2 - 4\frac{\partial \dot{k}}{\partial k}\frac{\partial \dot{q}}{\partial q}}\,\right]_{(k^*,\, q^*)} .$$

But (k*, q*) ∈ N, thus

$$\left.\frac{\partial \dot{q}}{\partial q}\right|_{(k^*,\, q^*)} = \left[-qf_1''(k_1)\frac{dk_1}{d\omega}\frac{\partial \omega}{\partial q}\right]_{(k^*,\, q^*)} > 0 ,$$

and

$$\frac{\partial \dot{k}}{\partial k} = \frac{-f_1(k_1)}{k_2 - k_1} - \lambda < 0 ,$$

since $k_2 > k_1$. The characteristic roots are real and opposite in sign and therefore the unique singular point (k*, q*) is a saddlepoint.[6]

As in Lecture I, the system (II.8) is closed by the appropriate transversality conditions

(II.9) $$k(0) = k_0, \quad q(T)e^{-\delta T}[k(T) - k_T] = 0, \quad \text{and } q(T) \geq 0 .$$

From Figure II.1 and the fact that the RHS of (II.8) are trivially Lipschitzian, we have the following Modified Golden Rule turnpike property for the trajectory $\{k^0(t) : t \in [0, T]\}$ that satisfies (II.8) and (II.9):

For T sufficiently large the Pontryagin capital-labor k^0 is close to k^* "most" of the time. More precisely:

Given $\epsilon > 0$ define the closed rectangular ϵ-neighborhood $N(\epsilon)$ of the Modified Golden Rule growth path by

$$N(\epsilon) \equiv \{(k, q) : |k - k^*| \leq \epsilon, |q - q^*| \leq \epsilon\}.$$

Then, for the unique Pontryagin growth path $\{k^0(t), q^0(t) : t \in [0, T]\}$ corresponding to the boundary parameters (k_0, k_T, T), there exist two finite times $0 \leq t_1 < \infty$ and $0 \leq t_2 < \infty$ such that $(k(t), q(t)) \in N(\epsilon)$ for $t_1 \leq t \leq t_2$.

Thus, for $T > t_1 + t_2$, $\{k^0(t), q^0(t) : t \in [0, T]\}$ spends time in $N(\epsilon)$. Furthermore as T becomes large the fraction of time spent in $N(\epsilon)$ approaches unity.

<u>The Case where Production of the Investment Good is More Capital Intensive than Production of the Consumption Good</u>

I shall briefly treat the reverse factor intensity case, i.e., the case where $k_1(\omega) > k_2(\omega)$ for $\omega > 0$. The analysis proceeds as before with the aid of the phase portrait described in Figure II.3. I leave the construction of the details of Figure II.3 as a "home problem."[7] It follows immediately that the Pontryagin trajectory possesses the Modified Golden Rule turnpike property. What about the case of equal factor intensities? This case, where $k_1(\omega) = k_2(\omega)$ is merely a generalization of the one-sector model presented in Lecture I and its analysis is almost identical to that of Lecture I.

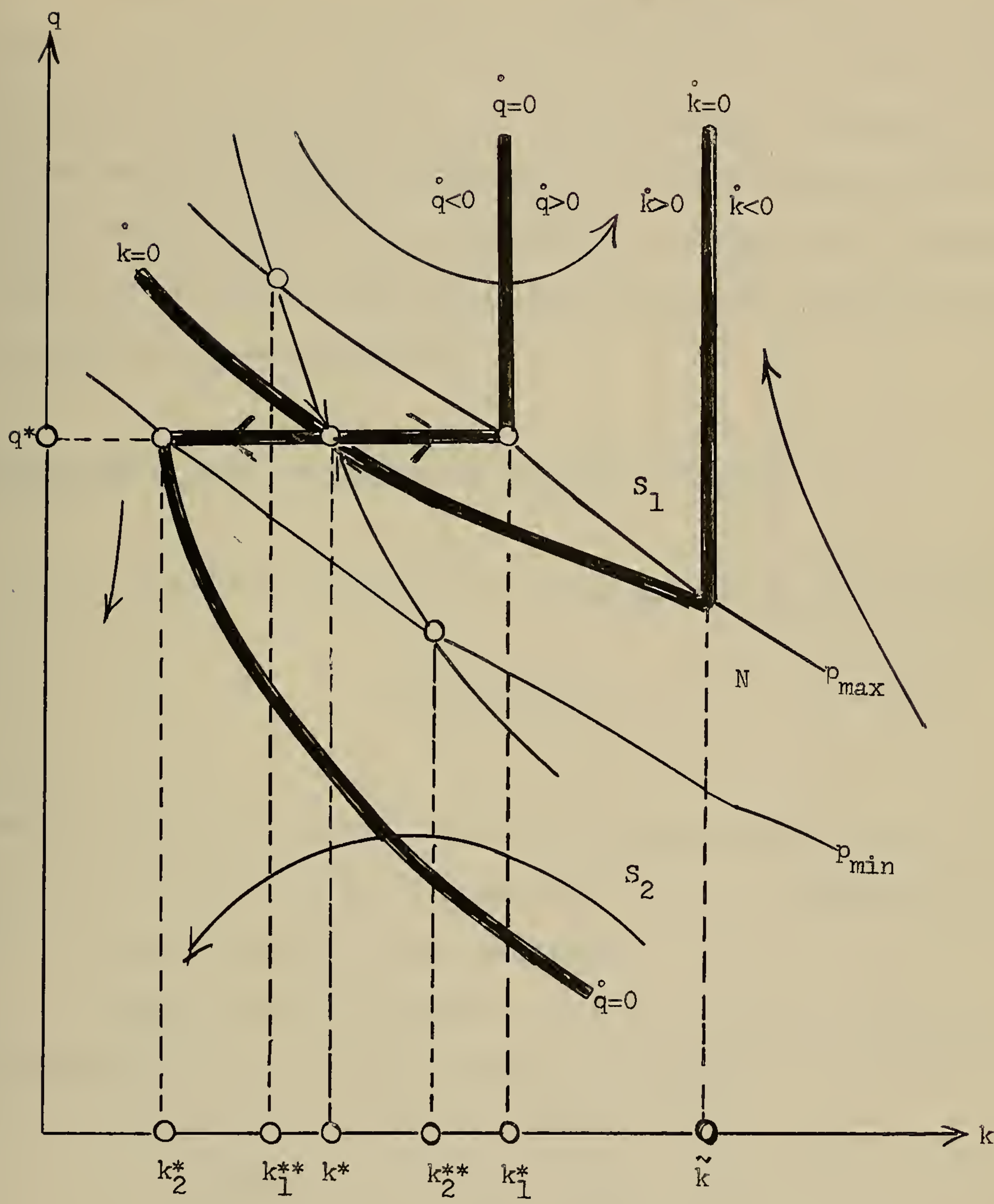


Figure II.3

What about the existence of optimal trajectories? For T sufficiently large, it can be shown that there exists one and only one trajectory satisfying the Maximum Principle and the Transversality Conditions. This follows because the stable branches of the (k^*, q^*) saddlepoint cover the entire positive k half-line. As in Lecture I, comparison of a Pontryagin trajectory with another feasible trajectory establishes that the necessary conditions are indeed sufficient.

The One-Sector Model with Nonlinear Objective Functional

If the linear objective functional is replaced by the maximand

$$\int_0^T U[c]e^{-\delta t}dt ,$$

then the one-sector analysis of Lecture I is significantly altered. The resulting turnpike property is no longer exact and the optimal savings ratio $s^0(t)$ is shown to be a continuous function.

Again, I will only sketch the analysis leaving the details as a "home problem."[8] The appropriate Hamiltonian form is

$$U[(1-s)f(k)]e^{-\delta t} + qe^{-\delta t}\{sf(k) - \lambda k\} .$$

The Maximum Principle requires for optimality that there exist a function $q(t)$ defined on $[0, T]$ and satisfying

$$\dot{q} = (\delta+\lambda)q - f'(k)\left\{U'[(1-s)f(k)]\right\}.$$

If we assume that $U'[c] > 0$ and $U''[c] < 0$ for $0 < c < \infty$ and that $U'[0] = \infty$, then the Maximum Principle implies that

$$U'[(1-s)f(k)] \geq q ,$$

and with equality for $s > 0$. The phase portrait that must be studied is supplied in Figure II.4. In the shaded region $U' > q$ and $s = 0$. (k^*, q^*) is a saddlepoint. The stable branch of the saddlepoint is shown entering once the (shaded) region of specialization. There is no particular reason for this to be the case. At any rate, however, the stable branches cover the positive k half-line. Thus, the problem of existence of an optimal control is trivial.

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In lectures I and II, I have tried to impart as wide a coverage of the applications of the Maximum Principle to the theory of optimal economic growth as is possible in such limited time. To gain this wide coverage, I have had to sacrifice precision. Even so, much has been left out. As a partial remedy I offer the following bibliography. The control theorist sampling these articles should gain an appreciation of the optimal growth problem. The economist may gain an insight into the power of the Maximum Principle by reading these articles while keeping accessible a copy of Pontryagin et al.

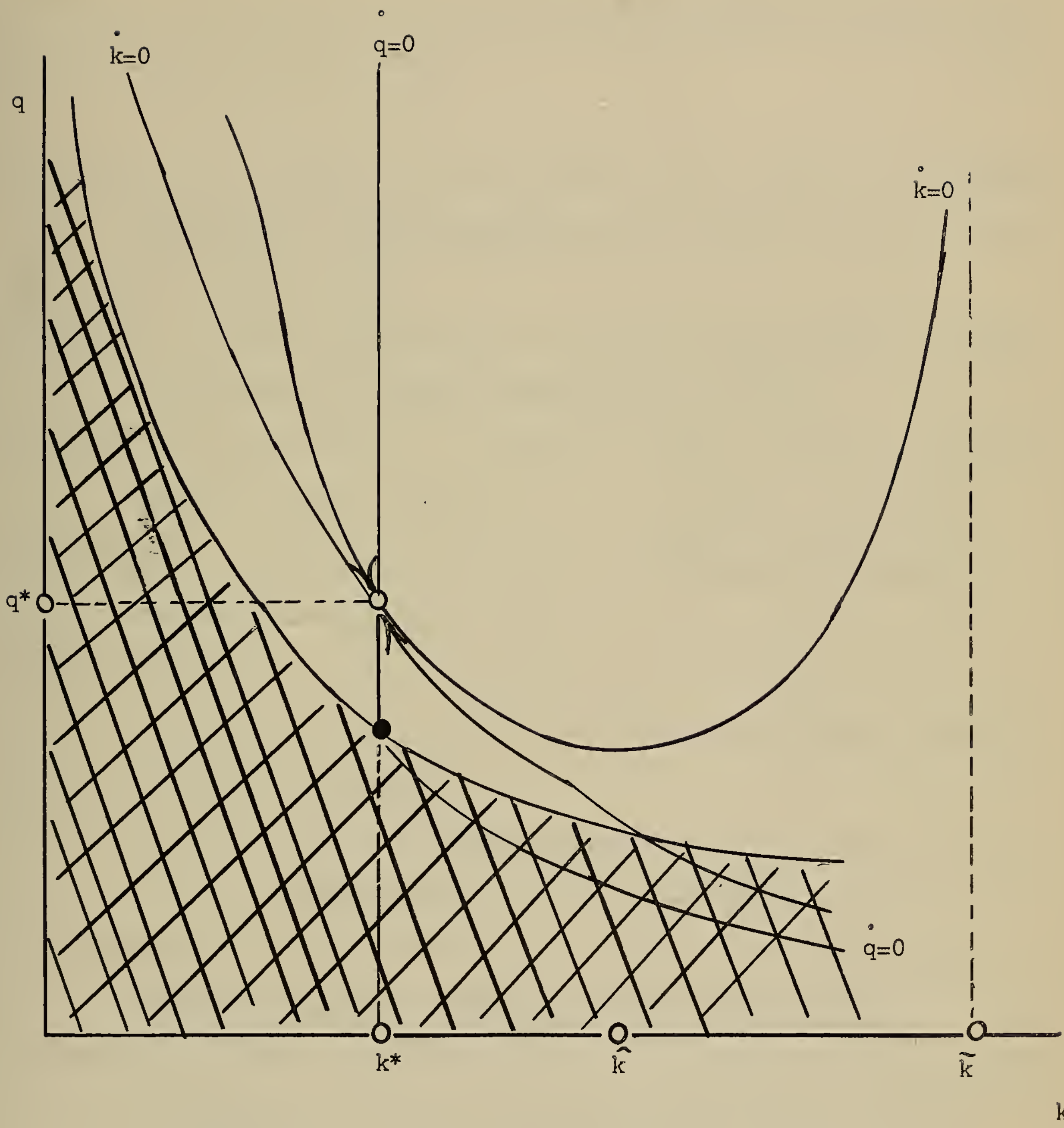

Figure II.4

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13. Stoleru, L. G., "An Optimal Policy for Economic Growth," Econometrica, Vol. 33 (April 1965), pp. 321-48.

14. Uzawa, H., "Optimum Technical Change in an Aggregative Model of Economic Growth," International Economic Review, Vol. 6, No. 1 (January 1965), pp. 18-31.

Footnotes to Lecture II

1. Meade, J. E., A Neoclassical Theory of Economic Growth, New York: Oxford University Press, 1961.

2. Uzawa, H., "On a Two-Sector Model of Economic Growth, II," Review of Economic Studies, Vol. 30 (1963), pp. 105-18.

3. In the terminology of Meade, the factors of production are assumed to be perfectly malleable.

4. A very similar problem was treated by H. Uzawa in "Optimal Growth in a Two-Sector Model of Capital Accumulation," Review of Economic Studies,

Vol. 31 (1964), 1-24. In an unpublished paper entitled "A Skeptical Note on Professor Uzawa's Two-Sector Model," Professor Wahidul Haque of the University of Toronto points out some difficulties in the Uzawa analysis. The difficulties are avoided in this lecture by using the techniques developed for a more general purpose in my paper "Optimal Programs of Capital Accumulation for an Economy in Which There is Exogenous Technical Change," in K. Shell (ed.), Essays on the Theory of Optimal Economic Growth, Cambridge, Mass. and London: M.I.T. Press, 1967.

5. As long as $k_1(\omega) \neq k_2(\omega)$ so that along the production possibility frontier $(\partial^2 c/\partial z^2) < 0$.

6. Cf. L. S. Pontryagin, Ordinary Differential Equations, pp. 246-54.

7. For the case $k_1 > k_2$, we have that $(\partial p/\partial \omega) < 0$. Further aid in the construction of Figure II.3 can be found in my "Optimal Programs of Capital Accumulation for an Economy in Which There is Exogenous Technical Change," op. cit.

8. Help in deriving the full analysis can be found in D. Cass, "Optimum Growth in an Aggregative Model of Capital Accumulation," Review of Economic Studies, Vol. 32 (July 1965), pp. 233-40.

LECTURE III

This will be a lecture about infinity. In particular, I will be concerned with the behavior of certain economic, optimizing, dynamical systems which are defined on $t \in [0, \infty)$.

I begin with an extension of the planning problem of Lecture I to the case where the planning horizon is infinite, i.e., the case where $T = \infty$. In Chapter IV, Pontryagin _et al._ treat the case where the functional to be optimized is given by an improper integral.[1] A Maximum Principle is derived for the case in which both the integral and the constraining differential equations are autonomous. In the notation of Pontryagin _et al._, they restrict themselves to the case where $\partial f^0/\partial t \equiv 0$ and $\partial f/\partial t \equiv 0$. Hence, Theorem 8[2/] is not immediately applicable to extensions of the planning problems treated in Lectures I and II, because when $\delta \neq 0$ our problem is "nonautonomous."

Nonetheless, it is a rather simple matter to extend the Maximum Principle to the case $T = \infty$ for the simple class of problems that have been treated in the first two lectures. The argument goes like this: Consider the extension of the problem of Lecture I. The maximand can always be rewritten as

$$\int_0^{t'} (1-s)f(k)e^{-\delta t}dt + \int_{t'}^{t''} (1-s)f(k)e^{-\delta t}dt + \int_{t''}^{\infty} (1-s)f(k)e^{-\delta t}dt ,$$

where $0 < t' < t'' < \infty$. If the program $\{k^0(t) : t \in [0, \infty)\}$ is optimal then the trajectory $\{k^0(t) : t \in [t', t'']\}$ must maximize

$$\int_{t'}^{t''} (1-s)f(k)e^{-\delta t}dt$$

subject to:

$$\overset{\circ}{k} = sf(k) - \lambda k ,$$

$$s \in [0, 1] ,$$

$$k(t') = k^0(t') ,$$

$$k(t'') = k^0(t'') .$$

In this case, Theorem 5 (page 65) of Pontryagin et al. applies. Therefore it is necessary for optimality that there exist a function q(t) defined on [t', t''] such that

$$\text{(MP)} \quad \begin{cases} \overset{\circ}{q} = (\delta+\lambda) - \gamma(q)f'(k), \\ \overset{\circ}{k} = s(q)f(k) - \lambda k , \\ \gamma(q) \equiv \max(1, q) , \\ s(q) \begin{cases} = 1, \text{ when } q > 1 , \\ \in [0, 1], \text{ when } q = 1, \\ = 0, \text{ when } q < 1 . \end{cases} \end{cases}$$

$$\text{(TC)} \qquad k(t') = k^0(t') \text{ and } k(t'') = k^0(t'') .$$

But on an optimal trajectory $\{k^0 : t \in [0, \infty]\}$ the Maximum Principle (MP) must hold for all (t', t'') such that $0 < t' < t'' < \infty$. Therefore,

$\{k^0 : t \in [0, \infty)\}$ must satisfy (MP). But what about tranversality conditions? If we require that $k(0) = k_0$ and $\lim_{t\to\infty} k(t) = k_\infty (k_\infty = 0$, or $k_\infty = k^*$, or $k_\infty = \tilde{k})$, then these two boundary conditions along with MP give a "sufficient system," i.e., a system with a unique solution. If, however, we impose no constraint on $\lim_{t\to\infty} k(t)$ then there does not seem to be any handy transversality condition available in the control theory literature.

Returning to the problem of maximizing $\int_0^\infty (1-s)f(k)e^{-\delta t}dt$ subject to $\dot{k} = sf(k) - \lambda k$, $s \in [0,1]$, and $k(0) = k_0$. This is a problem with "free right-hand endpoint." If $\delta \leq 0$ the problem has no solution, because clearly the improper integral is unbounded. Nonetheless, the problem can be solved for the case $\delta > 0$. Since (MP) is necessary, the phase portrait of Figure 2 (Lecture I) applies. A feasible path satisfying (MP) and $k(0) = k_0$ is the path that "proceeds directly" to $(q = 1, k = k^*)$ in finite time and maintains $(q = 1, k = k^*, s = s^* \equiv f(k^*)/\lambda k^*)$ forever after. On such a path $\lim_{t\to\infty} qe^{-\delta t} \equiv \lim_{t\to\infty} e^{-\delta t} = 0$. So that the limiting present value of the capital stock $\lim_{t\to\infty} e^{-\delta t}k^* = 0$. Using the sufficiency proof presented in Lecture I and replacing $q^0(T)e^{-\delta T}$ with $\lim_{t\to\infty} q^0(t)e^{-\delta t}$ and replacing $k^0(T)$ with $\lim_{t\to\infty} k^0(t) = k^*$, etc., reveals that this path satisfying the necessary conditions is indeed optimal. (Comparison variables $q(T)$, $k(T)$ etc. should be appropriately interpreted as limits inferior or limits superior of $q(t)$, $k(t)$, etc.)

For the problems of Lectures I and II the extension of the right-hand free endpoint condition that the present value of left-over capital, i.e., $q(T)e^{-\delta T}k(T) = 0 = q(T)e^{-\delta T}$, to $\lim_{t\to\infty} q(t)e^{-\delta t} = \lim_{t\to\infty} q(t)e^{-\delta t}k(k)$

when combined with (MP) provides a sufficient condition for optimality when $\delta > 0$.

The condition

$$\lim_{t\to\infty} q(t)e^{-\delta t}k(t) = 0$$

has been interpreted by several authors as meaning that the present social value of "left-over" capital be zero. The adjective "left-over" is very loosely used, of course, since the planning horizon is infinite. Some writers[3] have suggested that the above condition or simply the condition $\lim_{t\to\infty} qe^{-\delta t} = 0$ are necessary conditions due to Pontryagin et al. The argument was that the transversality condition derived for the finite horizon ($T < \infty$) free endpoint problem of Lectures I and II, namely the condition

$$q(T)e^{-\delta T} = 0$$

or since adding the constraint $k(T) \geq 0$ is not binding

$$q(T)e^{-\delta T}k(T) = 0$$

can be extended to the infinite horizon ($T = \infty$) case to read

$$\lim_{t\to\infty} q(t)e^{-\delta t} = 0 = \lim_{t\to\infty} k(t)q(t)e^{-\delta t} .$$

This argument is not in general correct. I will show by means of a counterexample that the above conditions are not necessary for optimality.

Consider the special case of no time preference with a criterion functional that is linear in consumption. Of course, $\int_0^\infty c\,dt$ is unbounded, so we resort to a trick which will give us a bounded maximand. Let $\hat{c}$ be golden rule consumption per capita, i.e., $\hat{c} \equiv f(\hat{k}) - \lambda\hat{k}$. Then consider the maximand

$$\int_0^\infty (c-\hat{c})dt \ .$$

This maximand is bounded from above since $\hat{c}$ is the maximum sustainable consumption per capita. Since there is no time discounting the Golden Rule and the Modified Golden Rule are identical: $\hat{c} = c^*$, $\hat{k} = k^*$, etc. (MP) tells us that the dynamical system as described by the phase portrait in Figure 2 (Lecture I) applies as a necessary condition. Consider the unique path satisfying (MP) <u>and</u> $\lim_{t\to\infty} q(t) = 1$ <u>and</u> $\lim_{t\to\infty} k(t) = k^*$. Call this program $\{(k^0(t), q^0(t)) : t \in [0,\infty)\}$. Let $\{(k(t), q(t)) : t \in [0,\infty)\}$ be any feasible comparison path. The class of feasible paths can be divided into four subclasses:

A. Paths on which $\lim_{t\to\infty} k(t) > \hat{k} = k^*$;

B. Paths on which $\lim_{t\to\infty} k(t) = \hat{k} = k^*$;

C. Paths on which $\lim_{t\to\infty} k(t) < \hat{k} = k^*$;

D. Paths on which $\lim_{t\to\infty} k(t)$ does not exist.

Programs on which (A) holds are dynamically inefficient (by the Phelps-Koopmans Inefficiency Theorem) and therefore are clearly dominated by the unique program satisfying (MP) and $\lim_{t\to\infty} k(t) = \hat{k} = k^*$. The value of the objective functional $\int_0^\infty (c-\hat{c})dt$ is finite on the path satisfying (MP) and $\lim_{t\to\infty} k(t) = \hat{k} = k^*$ and therefore this path is superior to paths satisfying (C), since on paths satisfying (C) $\int_0^\infty (c-\hat{c})dt = -\infty$. Paths satisfying (D) can be dominated since, by the Principle of Optimality, the optimal controller $s^0(t)$ is piecewise constant and on any subinterval of $[0, \infty)$ has at most three points of discontinuity.

Therefore, in the sufficiency proof that follows we can restrict ourselves to the class of comparison paths that tend to the golden-rule, i.e., paths for which $\lim_{t\to\infty} k(t) = \hat{k} = k^*$. By an argument analogous to the sufficiency proof of Lecture I, we have that

$$\int_0^\infty \{(c^0-\hat{c}) - (c-\hat{c})\} dt = \int_0^\infty (c^0-c)dt \geq \lim_{t\to\infty} \{k^0(t) - k(t)\} = 0 .$$

Thus on the optimal path, the discounted value of capital tends to a nonzero value; $\lim_{t\to\infty} q^0(t)k^0(t) = \hat{k} > 0$. This example is provocative in at least two ways. First, it whets the appetite for the search for a Transversality Condition for "free endpoint problems" in which the maximand is an improper integral. Second, it reminds the economist of the very special nature of economies with an infinite number of commodities. And the problem is posed in its most natural setting: intertemporal allocation.

I have searched the literature (and questioned my colleagues here at Varenna) and I have as yet been unable to find any positive answers in my search for such a transversality condition.

The following transversality condition could be proposed for cases in which the Maximum Principle is necessary:

$$\limsup_{t\to\infty,\ \tau\in[t,\infty)} \Psi(\tau)x(\tau) \geq \liminf_{t\to\infty,\ \tau\in[t,\infty)} \Psi(\tau)x^0(\tau),$$

where $x(\tau)$ is any feasible path of the state variables and $x^0(\tau)$ is the optimal (maximizing) path.[4]

That there exist efficient and indeed optimal trajectories along which the social discounted value of capital does not tend to zero should not come as a surprise after Professor Radner's lecture on efficient infinite programs. Radner has taught that on efficient programs, present values although representable by a linear functional need not be representable by an improper integral.[5] In the zero time discounting problem of this lecture since $\lim_{t\to\infty} q(t) = 1 \neq 0$, present discounted value of capital will have a representation as an improper integral plus an extra term.

Look at it this way: Assume that there is no population growth (just to make things easy) and that capital depreciates at the exponential rate $\mu > 0$. Consider the efficient (in the sense of Phelps-Koopmans) economy that is always in the Golden Rule state, so that $k(t) = k^*$ for $t \in [0, \infty)$. (Remember that for the above case with $\delta = 0$, $k^0(t) = \hat{k} = k^*$.) The rate of interest r is equal to the net marginal product of capital, $r = f'(\hat{k}) - \mu = 0$.

Consider the present discounted value PDV of the contract of renting capital over the period $[0, T]$, $T \in [0, \infty)$. At the end of the period the producer must return a unit of capital to its owner.

$$PDV = \int_0^T (f'-\mu)e^{-rt}dt + p(T)e^{-rT},$$

where p(T) is the market price of capital (equal to unity by the one-sector assumption) and $(f'-\mu)$ is the competitive repayment rate on such a contract. Since the above must hold for all $T \in [0, \infty)$, it must hold in the limit, i.e.,

$$PDV = \int_0^\infty (f'-\mu)e^{-rt}dt + \lim_{t\to\infty} p(t)e^{-rt} .$$

But by the Golden Rule assumption,

$$PDV = 0 + 1 = 1 .$$

We can think of PDV as the price of capital. By the one-sector assumption supply price of capital must be unity, so we have derived the zero-profit condition. As a quick check, consider the capital owner who rents out his capital but requires the producer to return that depreciated machine after T years.

$$PDV = \int_0^T f'e^{-\mu t}e^{-rt}dt + p(T)e^{-\mu T}e^{-rT}$$

$$= \int_0^T \mu e^{-\mu t}dt + e^{-\mu T}$$

$$= -e^{-\mu T} + 1 + e^{-\mu T} = 1 .$$

Or in the limit

$$\text{PDV} = \int_0^{\infty} \mu e^{-\mu t} dt + 0 = 1 .$$

Thus, in the above case, the zero-profit condition can be written as a linear functional in integral form or nonintegral form.

Another example of this type comes to us from the theory of the competitive corporation. Assume perfect foresight and that managers seek to maximize the value of the existing securities on behalf of the shareholders. If π is the price of a security, then

$$\pi(0) = \int_0^T De^{-R(t)} dt + \pi(T)e^{-R(T)} ,$$

where $D(t)$ is the dividend per share and the market discount factor $R(t) = \int_0^t r(s)ds$ with r the "market rate of interest." Since a shareholder is indifferent between income from dividends and from capital gains, the above equation can be instructively rewritten as

$$\pi(0) = \int_0^T (D + \dot{\pi})e^{-R(t)} dt + \pi(0)e^{-R(T)} .$$

These equations hold for all $T \in [0, \infty)$ and thus in the limit,

$$\pi(0) = \int_0^{\infty} De^{-R(t)} dt + \lim_{t \to \infty} \pi(t)e^{-R(t)} .$$

The price of a share is equal to the present discounted value of its stream of dividends plus an extra term. It is often assumed that the extra term vanishes. Why? If markets are in equilibrium, the return from holding a share must be exactly equal to "the market rate of interest," i.e.,

$$D/\pi + \dot{\pi}/\pi = r .$$

Hence when $0 < \lim_{t\to\infty} r(t) < \infty$, then $\lim_{t\to\infty} \pi(t)e^{-R(t)} = 0$ if asymptotically π is growing at a slower exponential rate than r. That is, PDV is representable by an improper integral in all cases but the one in which dividends per dollar are asymptotically zero. It is to be admitted that the firm with an asymptotically zero dividend-price ratio is a peculiar case. I have been unable, however, to find a force in the competitive model which will rule out such a possibility. The question is: Can a competitive economy live on its own dreams?

Footnotes for Lecture III

1. L. S. Pontryagin et al., op. cit., pp. 189-91.
2. L. S. Pontryagin et al., op. cit., p. 81.
3. Cf., e.g., D. Cass, "Optimum Growth in an Aggregative Model of Capital Accumulation," Review of Economic Studies, Vol. 32 (July 1965), pp. 233-40.
4. This conjecture is related to a conjecture made by Kenneth J. Arrow in private correspondence.
5. Actually Radner's lecture considered the case of a denumerably infinite number of commodities. He showed that present value although representable by a linear function need not be representable by an inner product of

prices and quantities.

LECTURE IV

In this lecture, I will present a descriptive model of heterogeneous capital accumulation that is due to Joseph Stiglitz and myself.[1] Although at one stage in the lecture I will apply Pontryagin's Maximum Principle, I am more interested in revealing the underlying dynamical system of the descriptive model and then relating it to the techniques developed in the earlier lectures.

The Shell-Stiglitz paper was inspired by the work of Frank Hahn. In two contributions[2] to the theory of heterogeneous capital accumulation, Hahn has studied the process of accumulation in an economy in which there is one consumption good and many different kinds of capital goods. Production functions are assumed to be Cobb-Douglas in the capitals and labor, there is no joint production and no intermediate good, and the labor force grows at a constant relative rate. It is assumed that factors are rewarded by their value marginal products and that all wages are consumed and all rentals are saved (and invested).

Hahn shows that if expectations about changes in the relative market prices of the capital goods are fulfilled, and if the capital, investment and consumption markets are required to be in momentary competitive equilibrium, then the balanced growth path (the path along which all capital stocks are growing at the rate of labor force growth and along which all relative prices remain constant) is uniquely determined. Next Hahn turns his attention to the problem of the determination of momentary competitive equilibrium (or the problem of what current outputs of the consumption and investment goods are consistent with the given current level of capital goods

stocks and the given current market prices for the consumption and investment goods). Although the existence of momentary competitive equilibrium is always assured, Hahn finds that only under restrictive (and difficult to interpret) assumptions is momentary competitive equilibrium output composition uniquely determined. From this he concludes that the system is not in general causal.

But even when momentary competitive equilibrium is uniquely determined, Hahn is able to show by means of a simple two-capital example that the system of differential equations (in the space of capitals measured in intensive units and their demand prices) is not globally stable in the sense of Lyapunov. That is, for certain initially given capital-labor ratios and initial assignment of prices for the respective capital goods, the economy does not proceed to the unique long-run balanced growth configuration. Thus Hahn concludes that when the complication of many capital goods is introduced into the model, the resulting process of development is qualitatively different from the development in the one-sector, one-capital model introduced by Solow. Hahn concludes further that since competitive equilibrium paths in the heterogeneous capital goods model do not tend to balanced growth, as is the case in certain planning models of heterogeneous capital accumulation,[3] something unsatisfactory in the process of capitalist development may have been uncovered.

<u>The Model</u>. Both the instability of long-run equilibrium and the lack of uniqueness of momentary equilibrium which Hahn observed in his

multi-sector Cobb-Douglas economy can be studied in a one-sector economy with two capital goods.[4] To make my analysis as close as possible to that of Hahn, I shall use the Cobb-Douglas production function to illustrate the more general problem. Then the production function can be written as

$$\text{(IV.1)} \qquad Y = F(K_1, K_2, L) = K_1^{\alpha_1} K_2^{\alpha_2} L^{1-\alpha_1-\alpha_2}, \; \alpha_1 > 0, \; \alpha_2 > 0, \; 1-\alpha_1-\alpha_2 > 0,$$

where K_i $(i = 1, 2)$ is the quantity of the ith capital good, L is labor, and Y is output. Letting $k_1 = (K_1/L)$ and $k_2 = (K_2/L)$ yields

$$\text{(IV.1')} \qquad Y = Lf(k_1, k_2) = Lk_1^{\alpha_1} k_2^{\alpha_2} .$$

Since the consumption good and both the investment goods are assumed to be produced by the same production function

$$\text{(IV.2)} \qquad Y = \overset{\circ}{K}_1 + \overset{\circ}{K}_2 + C + \mu K_1 + \mu K_2 = Z_1 + Z_2 + C ,$$

where $C \geq 0$ is consumption, $Z_i = \overset{\circ}{K}_i + \mu K_i \geq 0$ is gross investment in the ith capital good $(i = 1, 2)$, and $\mu > 0$ is the constant rate of exponential depreciation (assumed to be equal for the two capital goods). Once installed, the machines are bolted-down in the sense that they can no longer be transformed one into the other or into consumption goods.

Following Hahn, assume that all profits are invested and all wages consumed.

(IV.3) $$C = F_L L = (f - k_1 f_1 - k_2 f_2)L = (1 - \alpha_1 - \alpha_2)Y ,$$

where $F_L \equiv \partial F/\partial L$ and $f_i \equiv \partial f/\partial k_i$ $(i = 1, 2)$. Since the production supply prices of all three commodities are identical (since they are produced by the same production function), only the commodities with the highest market prices are produced. By assumption, both wages and profits are always positive, so that both consumption and investment are also always positive. Hence, taking the consumption good as numeraire yields

(IV.4) $$\max[p_1, p_2] = p_c \equiv 1,$$

where p_c, p_1, and p_2 are the prices of the consumption good, the first investment good, and the second investment good, respectively. The laws of capital accumulation are given by:

(IV.5) $$\begin{aligned} \dot{k}_1 &= \sigma(k_1 f_1 + k_2 f_2) - \lambda k_1 = z_1 - \lambda k_1 , \\ \dot{k}_2 &= (1-\sigma)(k_1 f_1 + k_2 f_2) - \lambda k_2 = z_2 - \lambda k_2 , \end{aligned}$$

where $z_i \equiv Z_i/L$ and σ is an upper semicontinuous correspondence in (p_2/p_1) given by

$$\sigma \begin{cases} = 1 & \text{if } p_2 < p_1 , \\ \in [0, 1] & \text{if } p_2 = p_1 , \\ = 0 & \text{if } p_2 > p_1 , \end{cases}$$

and λ is the sum of the rates of depreciation and population growth, both of which are assumed to be constant. That is, only the investment good with the higher price is produced.[5] Since wages are equal to consumption, rentals must equal gross investment. However, when $p_1 = p_2$, momentary equilibrium is not unique. A momentary equilibrium is an allocation of output among the consumption good and the two investment goods that satisfies our static behavioral relations (IV.2) - (IV.5), for currently given capital stocks, labor force, and prices. Nothing in this system determines the value of σ between 0 and 1 when $p_1 = p_2$. In Figure IV.1, I have drawn the production possibility frontier. I have then drawn, for the Cobb-Douglas case, the consumption line, which is the intersection of a plane parallel to the (Z_1, Z_2) plane and intersecting the C axis at a value equal to $(1 - \alpha_1 - \alpha_2)F$. If $p_1 > p_2$, it is clear that the economy operates at point A, while if $p_1 < p_2$, it operates at point B. But if $p_1 = p_2 = 1$, any point along the line AB is technologically and socially feasible.

Had there been curvature to the production possibility frontier, specifying the market price ratios would have uniquely specified the output bundle. The question is: what restrictions do we have to impose upon the production functions of a many-capital-goods, many-sector model to ensure that for any two capital goods i and j, $(\partial^2 \dot{Z}_i / \partial \dot{Z}_j^2)$ be nonzero along the production possibility frontier? This is essentially the question toward which much of Hahn's paper is directed.

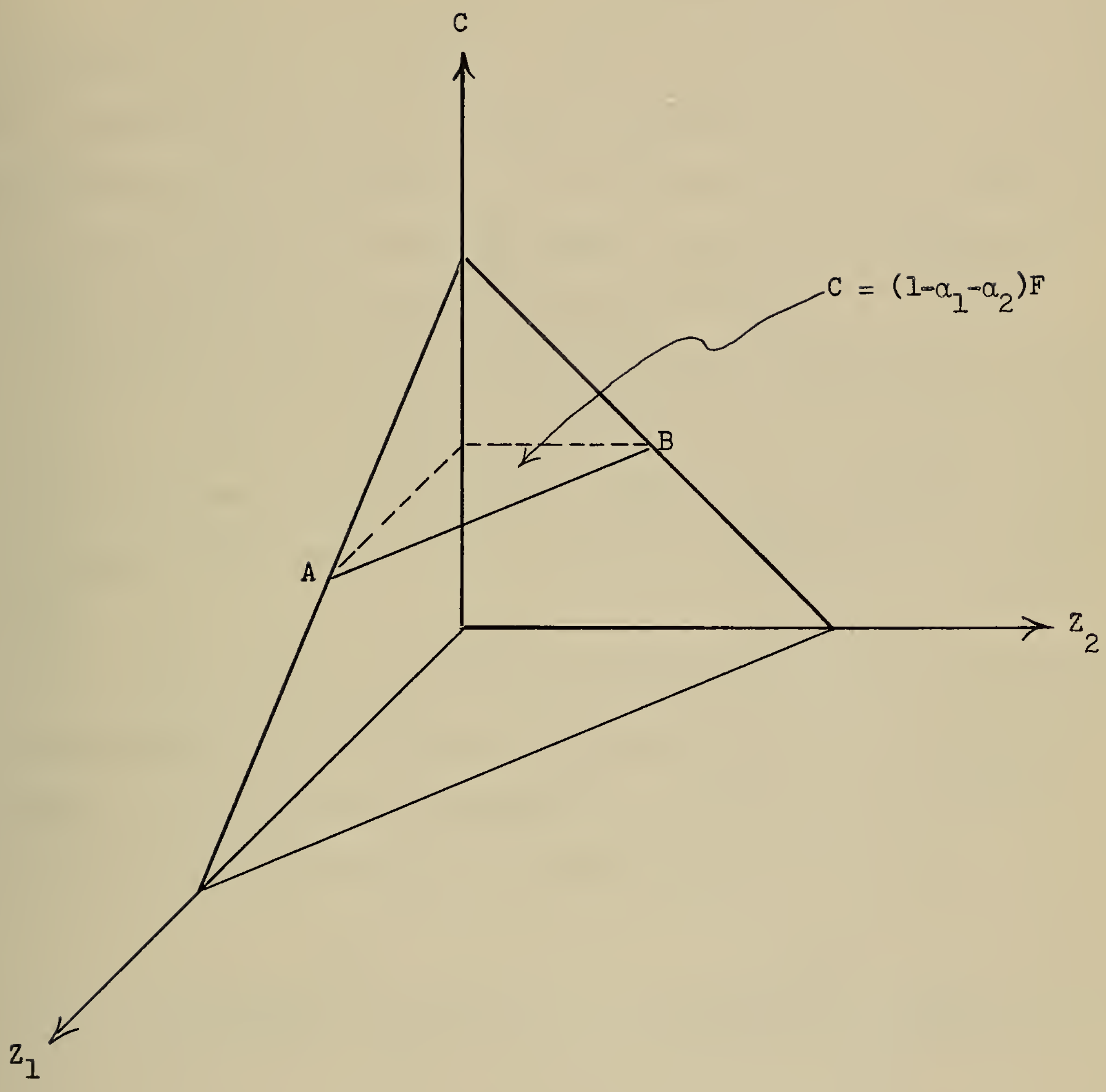

Figure IV.1

Thus far, I have discussed the markets equating demand and supply of consumption goods and demand and supply of investment goods. But I have yet to discuss the markets for already existing capital goods. We consider an individual at the beginning of the period with one unit of capital good one. He can hold this capital good and at the end of the period, the expected change in the value of his net worth will be $(1-\mu\Delta t)\hat{p}_1^{t+\Delta t} + r_1\Delta t - p_1^t$, where $\hat{p}_1^{t+\Delta t}$ is the expected price of his unit of K_1 at the end of the period; r_1 is the gross rentals per unit of time he gets from the ownership of a unit of K_1 (which under competition is equal to the marginal product of K_1) and μ is the rate of depreciation per unit of time.

Alternatively, he could have sold his unit of K_1, received p_1^t for it, purchased p_1^t/p_2^t units of K_2, and at the end of the period he would expect his net worth to have increased by $(p_1^t/p_2^t)[(1-\mu\Delta t)\hat{p}_2^{t+\Delta t} + r_2\Delta t - p_2^t]$. In equilibrium the expected returns are the same, i.e.,

$$\frac{(1-\mu\Delta t)\hat{p}_1^{t+\Delta t} + r_1\Delta t - p_1^t}{p_1^t} = \frac{(1-\mu\Delta t)\hat{p}_2^{t+\Delta t} + r_2\Delta t - p_2^t}{p_2^t} .$$

If individuals have short-run perfect foresight, so that expectations about price changes are realized, then

$$p_1^{t+\Delta t} = \hat{p}_1^{t+\Delta t} \quad \text{and} \quad p_2^{t+\Delta t} = \hat{p}_2^{t+\Delta t} .$$

Rearranging terms and taking the limit as $\Delta t \to 0$ yields

$$\dot{p}_1/p_1 + r_1/p_1 - \mu = \dot{p}_2/p_2 + r_2/p_2 - \mu \equiv \rho$$

where ρ is the rate of return, and where for a competitive economy r_1 and r_2 are simply f_1 and f_2.[6]

A special difficulty arises in the case where capital goods are freely disposable, since no capital goods price may fall below zero. That is, if $p_2 = 0$, $\dot{p}_2$ must be greater than or equal to zero. Hence, if marginal products are positive, the rate of return on K_2 is infinite, but the rate of return on K_1 is finite. Hence when the price ratio is zero, markets will not clear. Therefore we know that under short-run perfect foresight with short-run profit maximization

(IV.6) $$\dot{p}_1/p_1 + f_1/p_1 - \dot{p}_2/p_2 - f_2/p_2 = 0, \text{ or } p_1 p_2 = 0 .$$

Of course, if capital goods are not disposable, then the equation on the left in (IV.6) holds with equality for all p_1 and p_2. Whether or not paths which lead to $p_i = 0$ are consistent with a variety of institutional assumptions is the question treated by Shell-Stiglitz. However, since I am concentrating on the formal properties of the dynamical system, I will not discuss these very important issues in this lecture.

<u>Stationary Solutions</u>. Since the labor force growth is constant, balanced growth implies that k_1 and k_2 are constant. For k_1 and k_2 to remain constant, both of them must be produced, and hence both must have a price equal to the price of consumption goods (= 1). Moreover, in the balanced state both $\dot{p}_1/p_1$ and $\dot{p}_2/p_2$, must be zero. Hence from (IV.6), $f_1 = f_2$. From (IV.5), setting $\dot{k}_1 = 0 = \dot{k}_2$ gives $\lambda(k_1 + k_2) = f_1k_1 + f_2k_2 = f_1(k_1 + k_2)$, or $\lambda = f_1 = f_2$.

We now show that for the Cobb-Douglas case there exists a unique balanced growth path. Notice that $f_1 = f_2$ if and only if $k_1 = \alpha_1 k_2/\alpha_2$. For the Cobb-Douglas case, when $f_1 = f_2$

$$f_1 = (\alpha_1)^{1-\alpha_2} \alpha_2^{\alpha_2} k_1^{\alpha_1+\alpha_2-1} .$$

Differentiating the above with respect to k_1 shows that f_1 is decreasing in k_1 along the line $f_1 = f_2$. Therefore $f_1 = f_2 = \lambda$ for at most one value of the vector (k_1, k_2). Since along $f_1 = f_2$

$$\lim_{k_1 \to 0} f_1 = \infty \quad \text{and} \quad \lim_{k_1 \to \infty} f_1 = 0 ,$$

the solution (k_1^*, k_2^*) to the system $f_1 = \alpha_1 f/k_1 = f_2 = \alpha_2 f/k_2 = \lambda$ is uniquely determined.

Dynamic Analysis. As I have already noted, there are three different regimes in which the economy may find itself:

Regime I. $p_2 < p_1 = 1$, only capital good 1 is produced ($\sigma = 1$),

Regime II. $1 = p_2 > p_1$, only capital good 2 is produced ($\sigma = 0$),

Regime III. $p_1 = p_2 = 1$, σ is indeterminate, ($\sigma \in [0, 1]$).

The differential equations (IV.5) and (IV.6) reduce to:

Regime I		Regime II		Regime III	
(IV.7.I)	$\dot{k}_1 = \pi - \lambda k_1$	(IV.7.II)	$\dot{k}_1 = -\lambda k_1$	(IV.7.III)	$\dot{k}_1 = \sigma\pi - \lambda k_1$
(IV.8.I)	$\dot{k}_2 = -\lambda k_2$	(IV.8.II)	$\dot{k}_2 = \pi - \lambda k_2$	(IV.8.III)	$\dot{k}_2 = (1-\sigma)\pi - \lambda k_2$
(IV.9.I)	$\dot{p}_1 = 0$	(IV.9.II)	$\dot{p}_1 = p_1 f_2 - f_1$	(IV.9.III)	$\dot{p}_1 = \rho + \mu - f_1$
(IV.10.I)	$\dot{p}_2 = p_2 f_1 - f_2$	(IV.10.II)	$\dot{p}_2 = 0$	(IV.10.III)	$\dot{p}_2 = \rho + \mu - f_2$

where profits per capita $\pi = (\alpha_1 + \alpha_2)y$ with y denoting output per capita.

Although this is a system of four differential equations in k_1, k_2, p_1, and p_2, our simple assumptions about production and demand for consumption allow us to make a complete dynamic analysis of the system in (k_1, k_2) "phase" space. In Figure IV.2, I have drawn the ray OA along which $k_1 = k_2\alpha_1/\alpha_2$ (i.e., along which $f_1 = f_2$). Above OA, $f_2 > f_1$; below OA, $f_1 > f_2$.

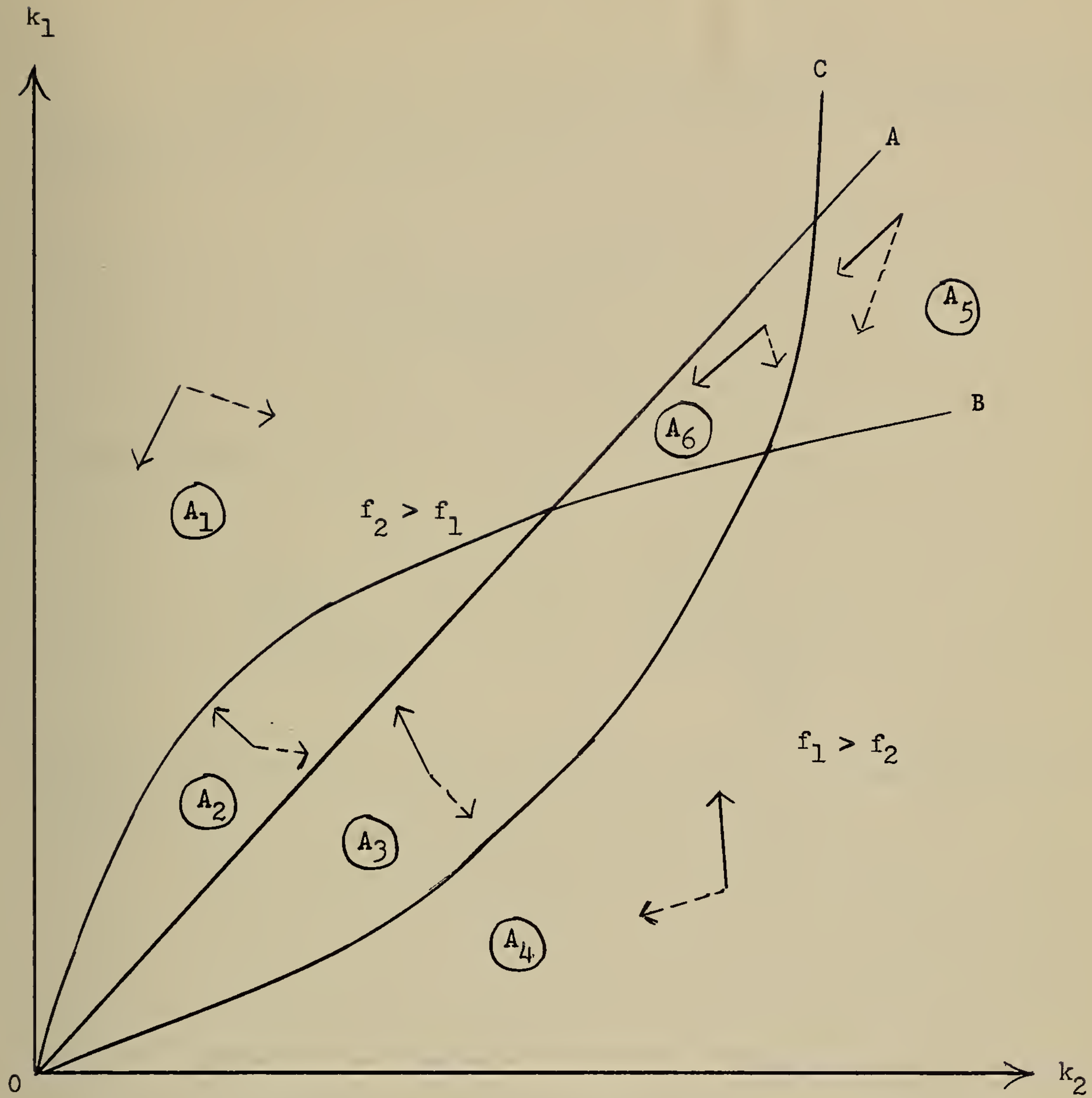

Figure IV.2

In Regime I, $\dot{k}_1 = 0$ along the curve denoted in Figure IV.2 by OB, which is the locus of points such that

$$k_1 = \left[\frac{(\alpha_1+\alpha_2)k_2^{\alpha_2}}{\lambda}\right]^{\frac{1}{1-\alpha_1}}$$

Differentiating yields

$$\left(\frac{dk_1}{dk_2}\right)_{\substack{\dot{k}_1=0\\ \sigma=1}} > 0 \qquad \text{and} \qquad \left(\frac{d^2k_1}{dk_2^2}\right)_{\substack{\dot{k}_1=0\\ \sigma=1}} < 0 .$$

Similarly for Regime II we can describe the locus of points such that $\dot{k}_2 = 0$ and this curve is denoted in Figure IV.2 by the curve OC.

Thus, Figure IV.2 is divided into six basic regions: A_1 which lies to the right of the k_1 (vertical) axis and above OA, OB, and OC; A_2 which lies above OA and below OB; A_3 which lies above OC and below OA and OB; A_4 which lies above the k_2 (horizontal) axis and below OC and OB; A_5 which lies above OB and below OC; A_6 which lies below OA and above OB and OC. The solid arrows indicate the direction of development in the respective regions when $\sigma = 1$ (Regime I). The dashed arrows indicate the direction of development when $\sigma = 0$ (Regime II).

So far, I have ignored the behavior of prices. Recall that in Regime I, $\dot{p}_2/p_2 = f_1 - f_2/p_2$. If the economy is in Regime I and above OA (i.e., $f_2 > f_1$), we know that $f_1 < f_2/p_2$ since $p_1 > p_2$. Thus, in this case

p_2 falls and as long as the economy is above OA it cannot switch to Regime II. It continues to specialize investment in the capital good with the lower marginal product--a clear instance of the Keynesian disparity between social and private returns due to capital gains. Similarly, if the economy is in Regime II and below OA, as long as it is below OA it cannot switch to Regime I.

We are now ready to put all this information to use for a full dynamic analysis. Consider for example an economy which begins initially in A_2. If the economy begins with $p_1 > p_2$ (i.e., in Regime I), it must remain in Regime I, so that the economy moves towards the curve OB, crosses it, and then moves towards the origin.

If initially $p_2 > p_1$, initially the economy is moving towards OA, but since $\dot{p}_1/p_1 = f_2 - f_1/p_1$ and $f_1 < f_2$, if p_1 is sufficiently large ($> f_1/f_2$), p_1 is rising. It is possible then that, before the economy gets to OA, p_1 becomes equal to p_2 (= 1).[7] But since $f_1 < f_2$, p_1 continues to rise, and we switch to Regime I. From then on the story follows as before. Alternatively, the economy can cross the ray OA with p_2 greater than p_1. The story for the economy in Regime II in A_3 is analogous to that of the economy in Regime I in A_2. The economy moves to OC, crosses it and proceeds to the origin.

One important case remains: The economy begins with $p_1 > p_2$, but $p_1 = p_2$ at exactly the moment that $f_1 = f_2$. The economy is then in Regime III in which momentary equilibrium is not unique.[8] There is a unique allocation σ^* which will enable the economy to move along the ray OA to the

steady-state aolution; we require $\overset{\circ}{k}_1/\overset{\circ}{k}_2$ to equal the slope of OA

$$\frac{\alpha_1}{\alpha_2} = \frac{\sigma^*\pi - \lambda k_1}{(1-\sigma^*)\pi - \lambda k_2} \quad ,$$

or

$$0 < \sigma^* = \frac{\alpha_1}{\alpha_1 + \alpha_2} < 1 \quad .$$

If σ deviates from σ^* for more than an infinitesimal length of time, clearly f_1 will no longer equal f_2. If, for instance, σ is too low, k_2 becomes slightly greater than $\alpha_2 k_1/\alpha_1$, i.e., f_1 becomes greater than f_2. Our price differential equation, for Regime III, is $\overset{\circ}{p}_2 - \overset{\circ}{p}_1 = f_1 - f_2 > 0$. Hence p_2 increases relative to p_1, and the economy moves into Regime II. From then on, the story is familiar.

But in the model as presented thus far there is no mechanism with only short-run perfect foresight by which σ can be maintained at σ^*.

The dynamic behavior for the economy with initial endowments in other regions can be analyzed in a similar manner.[9] For each initial assignment of the endowment vector (k_1, k_2), <u>there is one and only one assignment of initial prices</u> (p_1, p_2) <u>that allows the economy to proceed to long-run balanced growth</u>. Formally, we have shown that, if we assign $\sigma = \sigma^*$ in Regime <u>III, the unique balanced growth equilibrium is a saddlepoint in the</u> (k_1, k_2, p_1, p_2) <u>phase-space</u>. So far, there is no mechanism endogenous to the model which ensures that initial prices will be chosen so as to allow for long-run balanced growth. Paths not tending toward balanced growth tend to the

origin, and even along paths which allow for long-run balanced growth there is no mechanism to ensure that $\sigma = \sigma^*$ in Regime III.

Moreover, along paths not tending to balanced growth, the relative price of the two capital goods goes to zero in finite time. To see this, consider once again the economy in A_2 and Regime I. Defining $\beta = f_1/f_2$ and using (IV.10.1) yields

$$\overset{\circ}{p}_2 = (p_2\beta - 1)f_2 \quad \text{(IV.11)}$$

Observe that in this case β is declining through time. This is because β is a constant along every ray through the origin (where β decreases as the slope of the ray increases), and the path of development cuts every ray from the right. Also in this case, f_2 is increasing through time since

$$\frac{d \log f_2}{dt} = (1 - \alpha_1 - \alpha_2)\lambda + \frac{\alpha_1 \pi}{k_1} > 0 .$$

Therefore, from (IV.11), p_2 is falling at a rate faster than a constant absolute rate. Thus, for all paths not leading to long-run balanced growth, the price of the capital good with the higher marginal product goes to zero in finite time. Of course, if capital goods are not disposable, the price of the capital good with the higher marginal product will ultimately tend to minus infinity in all cases but the balanced growth case.

A Fancy Case. I now turn to a case which may be quite unrealistic in its assumptions. These assumptions will allow me to raise some interesting questions about the Maximum Principle and Transversality Conditions. The Shell-Stiglitz paper on which this lecture relies addresses itself to making judgments about the performance of capitalism under a variety of assumptions about the formation of price expectations and about market structure and institutions. At this time, however, I want to drop the burden of making such a weighty investigation and will turn to a more fanciful abstraction.

I assume that capital cannot be "disposed of." That is, there are no garbage dumps, although capital can be bought and sold in the market. I further assume that individuals possess perfect foresight about the entire future. I then ask whether or not there are institutions in the model capitalist economy "closing" the dynamical system just studied. So far, the initial capital-labor ratios can be thought of as historically given but where do initial prices come from?

Present Discounted Value Maximization. It might be thought that if producers maximize present discounted value of portfolios the model would be closed. This is not so. Integrating (IV.6) yields

$$p_i(0) = e^{-R(\tau)} p_i(\tau) + \int_0^{\tau} f_i e^{-R(t)} dt$$

for i = 1, 2, where $R(t) = \mu t + \int_0^t \rho(s)ds$. Hence provided differential equation (IV.6) holds everywhere the price is equal to PDV.[10]

The Capitalist as Bon Vivant. Although, following Hahn, workers are assumed to consume all, capitalists are assumed to save (and invest) in such a way as to maximize utility of their own lifetime consumption streams. The representative capitalist chooses $0 \le s(t) \le 1$ and $0 \le \sigma(t) \le 1$ in order to maximize

$$\int_0^\infty U[(1-s)\pi]e^{-\delta t}dt, \tag{IV.12}$$

where $\pi = r_1k_1 + r_2k_2$, and constrained by

$$\overset{\circ}{k}_1 = \frac{s\sigma\pi}{p_1} - \mu k_1, \quad \text{and} \tag{IV.13}$$

$$\overset{\circ}{k}_2 = \frac{s(1-\sigma)\pi}{p_2} - \mu k_2, \tag{IV.14}$$

where for ease of interpretation population growth is assumed to be zero, and $\delta > 0$ is the representative capitalist's subjective rate of time discount. Certainly the representative capitalist is free to consume out of capital gains (and in fact by selling assets) and even to shift his portfolio composition from one asset to the other without respecting the constraints: $s \in [0, 1]$ and $\sigma \in [0, 1]$. But the point is that if these constraints on s and σ are binding along any trajectory of market prices, then those market prices cannot be in equilibrium because some excess demands must be positive at those prices. The Hamiltonian H corresponding to the constrained maximization problem (IV.12) - (IV.14) can be written as

$$\text{(IV.15)} \qquad He^{\delta t} = U\left[(1-s)\pi\right] + \xi_1\left[\frac{s\sigma\pi}{p_1} - \mu k_1\right] + \xi_2\left[\frac{s(1-\sigma)\pi}{p_2} - \mu k_2\right] .$$

If $U'[0] = \infty$, then constrained maximization of (IV.15) with respect to s and σ yields

$$\text{(IV.16)} \qquad U'[(1-s)\pi] \geq \max\left(\frac{\xi_1}{p_1}, \frac{\xi_2}{p_2}\right),$$

with equality if $s > 0$. Furthermore, market equilibrium requires that $(\xi_1/p_1) = (\xi_2/p_2)$ or

$$\text{(IV.17)} \qquad \frac{\xi_1}{\xi_2} = \frac{p_1}{p_2} .$$

To see this, imagine a capitalist with given endowments K_1^0, K_2^0. If market prices of the capitals are p_1 and p_2, the capitalist's budget set in (K_1, K_2) space is defined by the budget line $p_1K_1 + p_2K_2 = p_1K_1^0 + p_2K_2^0$. Clearly, if (IV.17) does not hold the representative capitalist will specialize in holdings of his assets, which is contradictory of equilibrium in the used capital goods market. The same argument implies that (IV.16) must hold with equality in order for markets to clear. Along an equilibrium path, capitalists' own demand valuations must satisfy:

$$\text{(IV.18)} \qquad \dot{\xi}_1 = (\delta+\mu)\xi_1 - U'[(1-s)\pi]f_1,$$

$$\text{(IV.19)} \qquad \dot{\xi}_2 = (\delta+\mu)\xi_2 - U'[(1-s)\pi]f_2,$$

In order for the market equilibrium condition (IV.17) to hold at every instant of time it is required that

$$\text{(IV.20)} \qquad \frac{\dot{p}_1}{p_1} - \frac{\dot{p}_2}{p_2} = U'[(1-s)\pi]\left\{\frac{f_2}{\xi_2} - \frac{f_1}{\xi_1}\right\}$$

by (IV.18) and (IV.19). Or since (IV.16) holds with equality on competitive equilibrium paths, applying (IV.17) yields

$$\text{(IV.21)} \qquad \frac{\dot{p}_1}{p_1} - \frac{\dot{p}_2}{p_2} = \frac{f_2}{p_2} - \frac{f_1}{p_1} ,$$

which is, of course, the perfect foresight capital market clearing equation derived earlier. Along all competitive equilibrium trajectories (IV.21) holds, because if (IV.16) does not hold with equality there must be positive excess demands.

By an argument similar to that presented earlier in this lecture we know that on the path tending to balanced growth $\lim_{t\to\infty} p_1(t) = 1 = \lim_{t\to\infty} p_2(t)$. But on all other paths $\lim_{t\to\infty} p_i(t) = 1$ while $\lim_{t\to\infty} p_j(t) = -\infty$, for $i \neq j$.

Therefore on the path tending to balanced growth

$$\lim_{t\to\infty} \xi_1 e^{-\delta t} = 0 \lim_{t\to\infty} \xi_2 e^{-\delta t} .$$

On all other paths either $\xi_1 e^{-\delta t}$ or $\xi_2 e^{-\delta t}$ is unbounded from above.

As economists we know that nonpositive prices for assets forever bearing positive rentals are incompatible with utility-maximizing competitive equilibrium. To the control theorist: For this problem is the boundedness of $\xi_1 e^{-\delta t}$ and $\xi_2 e^{-\delta t}$ a necessary transversality condition?

Footnotes to Lecture IV

1. K. Shell and J. E. Stiglitz, "The Allocation of Investment in a Dynamic Economy," Quarterly Journal of Economics, Vol. 81 (November 1967).
2. F. H. Hahn, "Equilibrium Dynamics with Heterogeneous Capital Goods," Quarterly Journal of Economics, Vol. 80 (November 1966), pp. 633-46; "On the Stability of Growth Equilibrium," Memorandum, Institute of Economics, University of Oslo, April 19, 1966. Also see P. A. Samuelson, "Indeterminacy of Development in a Heterogeneous-Capital Model with Constant Saving Propensity," in K. Shell (ed.), Essays on the Theory of Optimal Economic Growth, Cambridge, Mass. and London: M.I.T. Press, 1967.
3. Cf., e.g., P. A. Samuelson and R. M. Solow, "A Complete Capital Model Involving Heterogeneous Capital Goods," Quarterly Journal of Economics, Vol. 70 (November 1956), pp. 537-62. Also cf. K. Shell, "Toward a Theory of Inventive Activity and Capital Accumulation," American Economic Review, Vol. 56 (May 1966), pp. 62-8.
4. Or equivalently, a three-sector model in which the capital intensities in all sectors are identical.

5. Notice that p_j has unit (C/K_j) $j = 1, 2$. Thus (p_2/p_1) has unit (K_1/K_2) which is the slope of the production possibility frontier in (Z_1, Z_2) space. By the one-sector assumption, the absolute value of the slope of the PPF is unity.

6. To see that the dimensions of the terms are consistent, write down the units of each as follows:

$$\frac{C/K_1 t}{C/K_1} + \frac{C/K_1 t}{C/K_1} - 1/t = \frac{C/K_2 t}{C/K_2} + \frac{C/K_2 t}{C/K_2} - 1/t \ .$$

7. Observe that in this case, at the point where $p_1 = p_2$, the system lacks uniqueness of momentary equilibrium. But this nonuniqueness lasts only for a moment. The amount of capital accumulation which occurs during that moment is infinitesimal, regardless of the value which σ takes on in that moment. The next moment, $p_2 > p_1$, and the economy's path is unaffected by what happens in the moment of nonuniqueness of equilibrium. Hence, although the economy lacks uniqueness of momentary equilibrium, it is not causally indeterminate.

8. It can be shown that for any given initial endowment there exists one and only one initial price which will get the economy to the OA ray at exactly the same moment that $p_1 = p_2 = 1$. First, we observe that if the economy remains in Regime I, it must, in finite time, cross OA, since

$$\frac{d \log(k_1/k_2)}{dt} = \pi/k_1 > 0 \ .$$

The right-hand side, in the region below OA, is clearly bounded away from zero, and hence in finite time, no matter what the initial value of k_1/k_2, the economy eventually reaches OA. The behavior of the real system (i.e., k_1, k_2) is independent of the particular prices chosen, provided that we remain in Regime I. Hence, the values of k_1 and k_2 along the path which goes from the initial value of (k_1, k_2) to a point on OA are determined at every point of time, and consequently, f_1 and f_2 are determined as functions of time alone. Since the Cobb-Douglas production function is analytic, the price differential equation satisfies the Lipschitz condition and hence the price differential equation, with the terminal condition $p_2(t^*) = 1$ where t^* is the time at which $f_1 = f_2$, has a unique backward solution.

9. If the economy begins in A_1 in Regime II, it either moves into A_2, from which point the story is familiar, or $p_2 = p_1$ before the economy gets to OG in which case it switches to Regime I. In Regime I and in A_1, the economy cannot switch to Regime II and must proceed toward the origin O. The behavior in A_3 and A_4 is symmetrical to that in A_1 and A_2. Observe from Figure IV.2 that an economy in A_5 or A_6 ultimately must proceed to A_1, A_2, A_3, or A_4.

10. It turns out that if $\lim_{t \to \infty} p_i(t)e^{-R(t)} \neq 0$ then $\lim_{t \to \infty} p_j(t)e^{-R(t)} = 0$, $(i \neq j)$. Therefore if a capitalist may not resell capital, $PDV_i(0) \neq PDV_j(0)$. But this is a strange restriction. Robert Hall in his M.I.T.

Ph.D. dissertation, "Essays on the Theory of Wealth," 1967, classifies economies in which for some i $\lim_{t\to\infty} p_i e^{-R(t)} \neq 0$ as "speculative boom" economies. This definition suggests that there is something basically "unsound" about such economies. However, we already know from Professor Radner's lectures and from my Lecture III, that there exist efficient competitive equilibrium economies for which the present value linear functional does not have an integral representation.